

HOMWORK 8, DUE WEDNESDAY MAY 28TH

1. Let $f: S^1 \rightarrow S^1$ be a continuous function which is not homotopic to the identity. Show that there is a point x such that $f(x) = x$ and a point y such that $f(y) = -y$.
2. We say that a subset A of $\mathbb{R}^n$ is **star convex** if there is a point a_0 such that if $a_1 \in A$ is any other point of A then the whole line segment from a_0 to a_1 is contained in A .
 - (a) Show that a convex set is always star convex but not vice-versa.
 - (b) Show that a star convex set is simply connected.
3. Let X be a topological space and let $A \subset X$ be a subspace. Let $r: X \rightarrow A$ be a retraction. If $a_0 \in A$ then show that

$$r_*: \pi_1(X, a_0) \rightarrow \pi_1(A, a_0).$$

is surjective.

4. If (X, x_0) and (Y, y_0) are two pointed spaces then show that

$$\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

5. A **topological group** G is a topological space, that is also a group, such that the group operations

$$\mu: G \times G \rightarrow G \quad \text{and} \quad i: G \rightarrow G$$

are continuous (here μ is the multiplication map and i is the inverse map).

Let G be a topological group with group operation $\star$ and identity x_0 . Let $\Omega(G, x_0)$ be the set of all loops based at x_0 .

Define a product on loops

$$\Omega(G, x_0) \times \Omega(G, x_0) \rightarrow \Omega(G, x_0)$$

by the rule

$$\gamma_0 \diamond \gamma_1 = \gamma_0(s) \star \gamma_1(s).$$

- (i) Show that $\Omega(G, x_0)$ becomes a group with this rule of multiplication.
- (ii) Show that $\pi_1(G, x_0)$ becomes a group with the inherited operation.
- (iii) Show that the two operations, $\diamond$ and $\cdot$, on $\pi_1(G, x_0)$ are the same (*Hint: Compute $(f \cdot c_{x_0}) \diamond (c_{x_0} \cdot g)$*).
- (iv) Show that $\pi_1(G, x_0)$ is abelian.