

HOMEWORK 7, DUE WEDNESDAY MAY 21ST

1. We recall the notion of a retraction from Math 190A. Let X be a topological space and let A be a subspace of X . We say that a continuous map $r: X \rightarrow A$ is a **retraction** if

$$r \circ i = \text{id}_A: A \rightarrow A.$$

If $f: X \rightarrow X$ is a function then we say that x is a **fixed point** if $f(x) = x$.

(i) Let X be the closed unit ball of radius one centred at the origin and let $A = S^{n-1}$ be the boundary of X .

Show that if there is a continuous map

$$f: X \rightarrow X$$

with no fixed points then there is a retraction

$$r: X \rightarrow A$$

(Let $U \subset X \times X$ be the complement of the diagonal. You may assume that the function l which sends $(x, y) \in U$ to the intersection of the ray from y to x with the unit sphere S^{n-1} is continuous).

(ii) By considering the induced maps on the homology of X and S^{n-1} conclude that every continuous map from X to X must have a fixed point.

2. Consider the antipodal map

$$a: S^n \rightarrow S^n \quad \text{given by} \quad a(x) = -x.$$

Show that if n is odd then a is homotopic to the identity map id_{S^n} (*Hint: Try the case $n = 1$ first*).

3. Let X and Y be topological spaces and let $f: X \rightarrow Y$ be a continuous map.

If there are two continuous maps

$$g: Y \rightarrow X \quad \text{and} \quad h: Y \rightarrow X$$

such that

$$g \circ f \sim \text{id}_X \quad \text{and} \quad f \circ h \sim \text{id}_Y$$

then show that f , g and h are homotopy equivalences.

4. Show that a retraction of a contractible space is contractible.

5. Let $f: S^m \rightarrow S^n$ be a continuous map.

(i) If f is not surjective then show that f is homotopic to a constant function.

(ii) If $m < n$ then show that f is homotopic to a constant function.
(*Hint: Use the simplicial approximation theorem.*)