

HOMEWORK 6, DUE WEDNESDAY MAY 14TH

1. Show that if $\mathcal{K}$ and $\mathcal{L}$ be are simplicial complexes and

$$h: |\mathcal{K}| \longrightarrow |\mathcal{L}|$$

is a continuous function and $\epsilon > 0$ then there are subdivisions $\mathcal{K}_0$ and $\mathcal{L}_0$ of $\mathcal{K}$ and $\mathcal{L}$ and a simplicial approximation f of h from $\mathcal{K}_0$ to $\mathcal{L}_0$ such that

$$||f(x) - h(x)|| < \epsilon,$$

for all $x \in |\mathcal{K}|$ (here $|\mathcal{L}| \subset \mathbb{R}^M$ and this is the standard norm).

2. Show that there is a category of chain complexes.

3. Let $f_\bullet$ and $g_\bullet$ be two chain maps from the chain complex $C_\bullet$ to the chain complex $D_\bullet$. Give the definition of a chain homotopy from f to g (here we drop the bullet subscript on f and g).

Show that

- (i) The relation $\sim$ on chain maps from $C_\bullet$ to $D_\bullet$ defined by $f \sim g$ if and only if f and g are chain homotopic is an equivalence relation.

- (ii) Composition of chain maps induces a well-defined map on chain homotopy classes (the equivalence classes of $\sim$).

- (iii) If f and g are chain homotopic then f_* and g_* induce the same map on homology.

We say that f and g are chain **chain homotopy inverses** of each other if

$$f \circ g: D_\bullet \longrightarrow D_\bullet \quad \text{and} \quad g \circ f: C_\bullet \longrightarrow C_\bullet$$

are both chain homotopic to the identity. In this case we say that f is a **chain equivalence**.

- (iv) If f and g are chain homotopy inverses then the induced maps on homology are inverses to each other.

- (v) If $C_\bullet$, $D_\bullet$ and $E_\bullet$ are chain complexes and

$$f: C_\bullet \longrightarrow D_\bullet \quad \text{and} \quad g: D_\bullet \longrightarrow E_\bullet$$

are two chain equivalences then the composition

$$g \circ f: C_\bullet \longrightarrow E_\bullet$$

is a chain equivalence.

4. Compute the homology of the closed ball of radius 1 in $\mathbb{R}^n$, the sphere S^n , the annulus, the Möbius band, the torus, the Klein bottle and the real projective plane.

Challenge Problems: (Just for fun)

5. Suppose we are given a commutative diagram

$$\begin{array}{ccccccccc}
 A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
 h_1 \downarrow & & h_2 \downarrow & & h_3 \downarrow & & h_4 \downarrow & & h_5 \downarrow \\
 B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5
 \end{array}$$

with exact rows.

- (i) If h_2 , h_4 are surjective and h_5 is injective then show that h_3 is surjective (ignore the first column).
- (ii) Give examples to show that if we drop any of the hypotheses about h_2 , h_4 and h_5 then h_3 need not be surjective.
- (iii) If h_2 , h_4 are injective and h_1 is surjective then show that h_3 is injective (ignore the last column).
- (iv) Give examples to show that if we drop any of the hypotheses about h_1 , h_2 and h_4 then h_3 need not be injective.
- (v) Conclude that if h_1 , h_2 , h_4 and h_5 are isomorphisms then h_3 is an isomorphism. This is known as the five Lemma.