

HOMEWORK 5, DUE WEDNESDAY MAY 7TH

1. A **homotopy** between two continuous maps f and g of two topological spaces X and Y

$$f: X \longrightarrow Y \quad \text{and} \quad g: X \longrightarrow Y$$

is a continuous function

$$F: X \times I \longrightarrow Y$$

such that

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = g(x).$$

- (i) Let f and g be two continuous map from X to $Y = \mathbb{R}^n$. Show that

$$F: X \times I \longrightarrow \mathbb{R}^n \quad \text{given by} \quad F(x, t) = (1 - t)f(x) + tg(x)$$

defines a homotopy from f to g .

- (ii) Let $\mathcal{K}$ and $\mathcal{L}$ be two simplicial complexes in $\mathbb{R}^n$ and let

$$h: |\mathcal{K}| \longrightarrow |\mathcal{L}|$$

be a continuous function. If f is a simplicial approximation of h then show that f and h are homotopic.

2. If σ is an m -simplex and we apply Barycentric subdivision to σ then how many simplices do we get if $m \leq 3$.

3. Let $\mathcal{K}$ be a complex and let x_0 be a point of $|\mathcal{K}|$.

- (i) Show that there is a subdivision $\mathcal{L}$ of $\mathcal{K}$ such that x_0 is a vertex of $\mathcal{L}$.

- (ii) Show that we can choose $\mathcal{L}$ so that every vertex of $\mathcal{L}$ is either x_0 or a vertex of $\mathcal{K}$.

4. If we are given two collections of sets $\mathcal{U}$ and $\mathcal{V}$ then we say that $\mathcal{V}$ **refines** $\mathcal{U}$ if for every $V \in \mathcal{V}$ we may find $U \in \mathcal{U}$ such that $V \subset U$.

A topological space X is said to have **finite covering dimension** if there is an integer m such for every open cover $\mathcal{U}$ of X there is an open cover $\mathcal{V}$, that is also a refinement of $\mathcal{U}$, such that no point of X lies in more than $m + 1$ elements of $\mathcal{V}$.

The **covering dimension** is the smallest integer m with this property.

- (i) Show that a discrete set has covering dimension zero.

- (ii) Show that $I = [0, 1]$ has covering dimension one.

- (iii) If X has covering dimension m then show that any closed subspace $Y \subset X$ has covering dimension m .

(iv) Show that any simplicial complex $\mathcal{K}$ has finite covering dimension and that the covering dimension m is at most the dimension of $\mathcal{K}$.

5. Here is another way to do Homework 3, Question 3. We have the same setup:

Let $\mathcal{K}$ be a simplicial complex and let σ and $\tau \in \mathcal{K}$ be two simplices, such that σ is not a proper face of a simplex in $\mathcal{K}$ and τ is a face of σ that is not a proper face of any other simplex in $\mathcal{K}$. Let $\mathcal{L} = \mathcal{K} \setminus \{\sigma, \tau\}$. Let

$$i: |\mathcal{L}| \longrightarrow |\mathcal{K}|.$$

be the simplicial map given by the inclusion of vertices.

(i) Construct a group homomorphism

$$j: C_m(\mathcal{K}) \longrightarrow C_m(\mathcal{L})$$

so that

$$j \circ i_* = \text{id}: C_m(\mathcal{L}) \longrightarrow C_m(\mathcal{L})$$

and

$$i_* \circ j: C_m(\mathcal{K}) \longrightarrow C_m(\mathcal{K}) \quad \text{and} \quad \text{id}: C_m(\mathcal{K}) \longrightarrow C_m(\mathcal{K}).$$

are chain homotopic.

(ii) Conclude that

$$i_*: H_m(\mathcal{L}) \longrightarrow H_m(\mathcal{K})$$

is an isomorphism, for all j .

Challenge Problems: (Just for fun)

6. Find the number of simplices in the Barycentric subdivision of an m -simplex σ .