

HOMEWORK 4, DUE WEDNESDAY APRIL 30TH

1. Let F be a functor from a category $\mathcal{C}$ to a category $\mathcal{D}$. Show that if X and Y are isomorphic objects of $\mathcal{C}$ then $F(X)$ and $F(Y)$ are isomorphic objects of $\mathcal{D}$.

2. Let $\mathcal{C}$ be a locally small category, so that given any two objects X and Y of $\mathcal{C}$ the collection $\text{Hom}(X, Y)$ of morphisms X to Y is a set.

Let A be an object of $\mathcal{C}$.

(i) Define a covariant functor h_A from $\mathcal{C}$ to the category of sets, denoted Sets, as follows:

To every object X we let $h_A(X) = \text{Hom}(A, X)$ be the set of all morphisms from A to X . To every morphism $f: X \rightarrow Y$ we define a function

$$h_A(f): \text{Hom}(A, X) \rightarrow \text{Hom}(A, Y) \quad \text{by the rule} \quad h_A(f)(g) = f \circ g.$$

Check that h_A is indeed a functor.

(ii) Define a contravariant functor h^A from $\mathcal{C}$ to Sets, as follows:

To every object X we let $h^A(X) = \text{Hom}(X, A)$ be the set of all morphisms from X to A . To every morphism $f: X \rightarrow Y$ we define a function

$$h^A(f): \text{Hom}(Y, A) \rightarrow \text{Hom}(X, A) \quad \text{by the rule} \quad h^A(f)(g) = g \circ f.$$

Check that h^A is indeed a functor.

3. Let $\mathcal{K}$ be the simplicial complex determined by an n -simplex σ .

(i) Show that $\mathcal{K}$ is a cone over a simplicial complex $\mathcal{M}$.

(ii) Show that the reduced homology groups of $\mathcal{K}$ are all zero,

$$\tilde{H}_m(\mathcal{K}) = 0.$$

(iii) Compute the homology of the $(n-1)$ -skeleton $\mathcal{L}$ of $\mathcal{K}$.

4. Let $\mathcal{K}$ be a simplicial complex. Suppose that we are given two points w_1 and w_2 such that the support of the two cones $\text{Cone}_{w_1}(\mathcal{K})$ and $\text{Cone}_{w_2}(\mathcal{K})$ intersects only along $|\mathcal{K}|$.

(i) Show that

$$S(\mathcal{K}) = \text{Cone}_{w_1}(\mathcal{K}) \cup \text{Cone}_{w_2}(\mathcal{K})$$

is a simplicial complex. $S(\mathcal{K})$ is called the **suspension** of $\mathcal{K}$.

(ii) If $\mathcal{K}$ is the 1-skeleton of a 2-simplex then show that $S^n(\mathcal{K})$ is a triangulation of the sphere S^{n+1} , where the simplicial complex $S^n(\mathcal{K})$

is defined recursively by the rule

$$S^n(\mathcal{K}) = \begin{cases} \mathcal{K} & \text{if } n = 0 \\ S(S^{n-1}(\mathcal{K})) & \text{otherwise.} \end{cases}$$

(iii) Define

$$\phi: C_m(\mathcal{K}) \longrightarrow C_{m+1}(S(\mathcal{K})) \quad \text{by the rule} \quad \phi(\alpha) = \langle w_1, \alpha \rangle - \langle w_2, \alpha \rangle.$$

Show that ϕ induces a group homomorphism

$$\phi_*: H_m(\mathcal{K}) \longrightarrow H_{m+1}(S(\mathcal{K})).$$

(iv) Going back to the notion of (ii), compute the homology of $S^n(\mathcal{K})$, *assuming* that

$$\phi_*: \tilde{H}_m(S^{n-1}\mathcal{K}) \longrightarrow \tilde{H}_{m+1}(S^n(\mathcal{K})),$$

is an isomorphism, for all $m > 0$.