

HOMEWORK 3, DUE WEDNESDAY APRIL 23RD

1. (i) Write down triangulations of the annulus and of the circle, $\mathcal{K}$ and $\mathcal{L}$, such that there is a simplicial map

$$f: |\mathcal{K}| \longrightarrow |\mathcal{L}|.$$

- (ii) Use f to compute the homology of $\mathcal{K}$.

2. Let $\mathcal{K}$ be a simplicial complex and let $\sigma \in \mathcal{K}$ be a simplex that is not a proper face of any simplex in $\mathcal{K}$. Let $\mathcal{L} = \mathcal{K} \setminus \{\sigma\}$.

- (i) Show that $\mathcal{L}$ is a simplicial complex.

- (ii) Show that the inclusion of vertices naturally induces a simplicial map

$$i: |\mathcal{L}| \longrightarrow |\mathcal{K}|.$$

- (iii) Now suppose that σ is an n -simplex.

Show that the map

$$i_*: H_j(\mathcal{L}) \longrightarrow H_j(\mathcal{K})$$

is an isomorphism, for $j \neq n - 1$ and n .

- (iv) Show that there is a group homomorphism

$$\pi: \mathbb{Z} \longrightarrow H_{n-1}(\mathcal{L})$$

induced by the map

$$\mathbb{Z} \longrightarrow C_{n-1}(\mathcal{L})$$

which sends 1 to the $(n - 1)$ -chain

$$\partial_n \sigma \in C_{n-1}(\mathcal{L}).$$

- (v) Show that there is a group homomorphism

$$\phi: H_n(\mathcal{K}) \longrightarrow \mathbb{Z}$$

which induces an exact sequence

$$0 \longrightarrow H_n(\mathcal{L}) \xrightarrow{i_*} H_n(\mathcal{K}) \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\pi} H_{n-1}(\mathcal{L}) \xrightarrow{i_*} H_{n-1}(\mathcal{K}) \longrightarrow 0.$$

Exact means that the kernel of each map is the image of the preceding map (somewhat fancifully, this is a chain complex whose homology is zero).

3. Let $\mathcal{K}$ be a simplicial complex and let σ and $\tau \in \mathcal{K}$ be two simplices, such that σ is not a proper face of a simplex in $\mathcal{K}$ and τ is a face of σ that is not a proper face of any other simplex in $\mathcal{K}$. Let $\mathcal{L} = \mathcal{K} \setminus \{\sigma, \tau\}$.

- (i) Show that $\mathcal{L}$ is a simplicial complex.

(ii) Show that the inclusion of vertices naturally induces a simplicial map

$$i: |\mathcal{L}| \longrightarrow |\mathcal{K}|.$$

(iii) Show that the map

$$i_*: H_j(\mathcal{L}) \longrightarrow H_j(\mathcal{K})$$

is an isomorphism, for all j .

4. Pick triangulations of the torus and the real projective plane and use the previous two questions to compute their homology.