

HOMEWORK 2, DUE WEDNESDAY APRIL 16TH

1. Define a complex $\mathcal{K}$ as follows. Take the unit square I^2 and divide it into four triangles by adding the two diagonals. Now remove the interior of the triangle with side contained in $x = 1$. The resulting topological space is the support of a simplicial complex $\mathcal{K}$ with three 2-simplices, in an obvious way.

Compute the homology groups of $\mathcal{K}$.

2. Pick a triangulation of the Möbius band and compute its homology groups.

3. Let $\mathcal{K}$ be the 2-skeleton of the simplicial complex associated to a 3-simplex. Compute its homology groups.

4. Pick a triangulation of the torus and compute its homology groups.

5. Pick a triangulation of the real projective plane and compute its homology groups. What is different in this case?

You may want to resort to using a computer algebra package for some of these computations.

Challenge Problems: (Just for fun)

6. The i th Betti number $b_i(\mathcal{K})$ of a simplicial complex $\mathcal{K}$ is the rank of the i th homology group $H_i(\mathcal{K})$ (recall that any finitely generated abelian group is isomorphic to $\mathbb{Z}^r \oplus T$, where T is finite. The rank r is simply the number of factors of $\mathbb{Z}$).

The **Euler characteristic** is then the alternating sum

$$b_0 - b_1 + b_2 - b_3 + \dots$$

(i) Show that the Euler characteristic is the alternating sum of the ranks of the chain groups.

(ii) Compute the Euler characteristic of the simplicial complexes in questions 1-5 in two different ways. First, as the alternating sum of the Betti numbers and secondly as the alternating sum of the ranks of the chain groups.