

HOMEWORK 1, DUE WEDNESDAY APRIL 9TH

1. Let $p_0, p_1, \dots, p_m$ be points of $\mathbb{R}^n$.
 - (a) If $m > n$ then show that the following are equivalent:
 - (i) The points $p_0, p_1, \dots, p_m$ are in linear general position.
 - (ii) No subset of $n + 1$ points lies in a hyperplane (an affine subspace of dimension $n - 1$).
 - (iii) Any collection of n vectors $v_1, v_2, \dots, v_n$ is linearly independent, where

$$q_0, q_1, \dots, q_n$$

are any $(n + 1)$ points taken from $p_0, p_1, \dots, p_m$ and

$$v_i = q_i - q_0.$$

- (b) If $m \leq n$ then show that the following are equivalent:
 - (i) The points $p_0, p_1, \dots, p_m$ are in linear general position.
 - (ii) The span $p_0, p_1, \dots, p_m$ of the points is an affine linear space of dimension m .
 - (iii) The vectors $v_1, v_2, \dots, v_m$ are linearly independent, where

$$v_i = p_i - p_0.$$

2. Show that the convex hull of a subset $X \subset \mathbb{R}^n$ makes sense (that is, prove that there is a smallest convex set C containing X).
3. How many faces does an n -simplex have? Prove your answer is correct.
4. Let $\mathcal{K}$ be a simplicial complex and let $k \geq 0$ be a non-negative integer. The k -**skeleton** $\mathcal{L}$ of $\mathcal{K}$ is the subset of all simplices of $\mathcal{K}$ of dimension at most k . Show that $\mathcal{L}$ is a simplicial complex.
5. (a) Write down triangulations of the sphere S^2 , the torus $S^1 \times S^1$, the Klein bottle and the real projective plane $\mathbb{RP}^2$.
 (b) In each case compute the number of vertices (or 0-simplices), edges (or 1-simplices) and faces (or 2-simplices) and the number

$$\chi = v - e + f.$$

6. An **abstract simplicial complex** consists of a finite set V_X , called the *vertices*, and a subset X of the power set of V_X (so that X is a collection of subsets of the vertices), called the *simplices*, such that if $\sigma \in X$ and $\tau \subset \sigma$ then $\tau \in X$.

A morphism

$$f: (V_X, X) \longrightarrow (V_Y, Y)$$

between abstract simplicial complexes is a function

$$f: V_X \longrightarrow V_Y$$

such that

$$f(\sigma) \in Y,$$

whenever $\sigma \in X$.

(i) Show that this gives us a category.

(ii) Let $\mathcal{K}$ be a simplicial complex. Let V_X be the set of 0-simplices of $\mathcal{K}$. A subset $\Sigma \subset V_X$ is in X if and only if there is a simplex $\sigma \in \mathcal{K}$ whose vertices are the elements of Σ .

Show that (V_X, X) is an abstract simplicial complex. We will call (V_X, X) the **abstraction** of $\mathcal{K}$.

(iii) If two simplicial complexes $\mathcal{K}$ and $\mathcal{L}$ have isomorphic abstractions (V_X, X) and (V_Y, Y) then show that $\mathcal{K}$ and $\mathcal{L}$ are homeomorphic.

Challenge Problems: (Just for fun)

(iv) Show that there are infinitely many points in $\mathbb{R}^n$ in linear general position.

(v) If (V_X, X) is an abstract simplicial complex and the largest cardinality of an element of X is $n + 1$ (in other words the dimension of any simplex is at most n) then there is a simplicial complex $\mathcal{K}$ in $\mathbb{R}^{2n+1}$ whose abstraction is isomorphic to (V_X, X) .