MODEL ANSWERS TO THE NINTH HOMEWORK

1. (i) We just have to show that T is non-empty, closed under addition and scalar multiplication.
$0 \in T$ as $1 \cdot 0 = 0$. Thus T is non-empty.
Suppose that m and n belong to T. Then we may find non-zero r and s such that $rm = 0$ and $sn = 0$. Note that $rs \neq 0$ as R is an integral domain and that

$$\begin{aligned}(rs)(m+n) &= (rs)m + (rs)n \\ &= (sr)m + (rs)n \\ &= s(rm) + r(sn) \\ &= s0 + r0 \\ &= 0 + 0 \\ &= 0.\end{aligned}$$

Thus $m + n \in T$. It follows that T is closed under addition.
Now suppose that $m \in T$ and $s \in R$. By assumption we may find $r \in R$ such that $rm = 0$ and $r \neq 0$. We have

$$\begin{aligned}r(sm) &= (rs)m \\ &= (sr)m \\ &= s(rm) \\ &= s0 \\ &= 0.\end{aligned}$$

Thus $sm \in T$. Thus T is closed under scalar multiplication.
It follows that T is indeed a submodule.
(ii) (a) The whole of $\mathbb{Q}/\mathbb{Z}$.
Indeed, if $a/b \in \mathbb{Q}$ then

$$b(a/b) = a \in \mathbb{Z}.$$

Thus the left coset

$$a/b + \mathbb{Z} \in T.$$

(b) $\mathbb{Q}/\mathbb{Z}$.
Indeed, we have already seen that

$$T \subset \mathbb{Q}/\mathbb{Z}.$$

Suppose that $r \in \mathbb{R}$ and $a \in \mathbb{Z}$ such that

$$ar = b \in \mathbb{Z}.$$

Solving for r gives

$$r = \frac{b}{a} \in \mathbb{Q}.$$

(c) 0.
Indeed, if $r \in \mathbb{R}$ and $a \in \mathbb{Z}$ such that

$$ar = b/c \in \mathbb{Q}$$

then solving for r gives

$$r = b/(ca) \in \mathbb{Q}.$$

Thus r is a rational.
(iii) (a) Yes, this is clear.
(b) No. $\mathbb{Q}$ is certainly not cyclic. Suppose that $q = a/b$ and $r = c/d \in \mathbb{Q}$. Then

$$(ad)r - (bc)q = ac - ac = 0.$$

Thus no two elements of $\mathbb{Q}$ generate a free subgroup of rank 2. But any free group of rank at least two contains a free subgroup of rank 2.
(c) No. If $q_1, q_2, \dots, q_k$ belong to $\mathbb{Q}$ and $q_i = a_i/b_i$ then the subgroup they generate belongs to the group generated by $1/b$ where $b = b_1 b_2 \dots b_k$. But $1/(b+1) \in \mathbb{Q}$ is not of this form.
2. (i) By the classification of finitely generated modules over a PID M is isomorphic to $F \oplus T$. In this case $M/T \simeq F$. It is clear that F is maximal with this property.
(ii) This is again immediate from the classification, just consider the map

$$F \longrightarrow T \oplus F \qquad \text{given by} \qquad f \longrightarrow (0, f).$$

(iii) Consider $\mathbb{Z} \oplus \mathbb{Z}_2$. Then $F \simeq \mathbb{Z}$. There are two inclusions

$$\mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}_2 \qquad \text{given by} \qquad n \longrightarrow (n, 0) \qquad \text{and} \qquad n \longrightarrow (n, n).$$

The image of the first is the subgroup $\langle (1,0) \rangle$ image of the second is the subgroup $\langle (1,1) \rangle$. Both subgroups are isomorphic to $\mathbb{Z}$.
3. (i) We start with

$$A = \begin{pmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{pmatrix}.$$

Since one entry is 2 and there are odd entries, the gcd of the entries of A is 1. If we take the second row multiply by -2 and add it the first row we get

$$\begin{pmatrix} -8 & -10 & -1 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{pmatrix}.$$

Now we if multiply the first row by -1 we get

$$\begin{pmatrix} 8 & 10 & 1 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{pmatrix}.$$

Now we swap the first and third columns to get

$$\begin{pmatrix} 1 & 10 & 8 \\ 4 & 2 & 2 \\ 15 & 6 & 6 \end{pmatrix}.$$

Now the entry in the top left hand corner is the gcd of the entries of A. We now eliminate the entries in the first column and the first row. We take the first row, multiply by -4 and add it to the second row.

$$\begin{pmatrix} 1 & 10 & 8 \\ 0 & -38 & -30 \\ 15 & 6 & 6 \end{pmatrix}.$$

Now we take the first row, multiply by -15 and add it to the third row

$$\begin{pmatrix} 1 & 10 & 8 \\ 0 & -38 & -30 \\ 0 & -144 & -114 \end{pmatrix}.$$

Now we use the first column to eliminate the entries in the first row

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -38 & -30 \\ 0 & -144 & -114 \end{pmatrix}.$$

We multiply the second and third rows by -1

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 38 & 30 \\ 0 & 144 & 114 \end{pmatrix}.$$

We focus on the matrix

$$\begin{pmatrix} 38 & 30 \\ 144 & 114 \end{pmatrix}.$$

Every entry is even. So the gcd is even. The first entry is $38 = 2 \cdot 19$. The second entry is coprime to 19. So the gcd is 2.

If we temporarily divide every entry by 2 we get

$$\begin{pmatrix} 19 & 15 \\ 72 & 57 \end{pmatrix}.$$

The gcd is now one. If we subtract the second column from the first we get

$$\begin{pmatrix} 4 & 15 \\ 15 & 57 \end{pmatrix}.$$

If we take the first column, multiply by -4 and add it to the second column then we get

$$\begin{pmatrix} 4 & -1 \\ 15 & -3 \end{pmatrix}.$$

Now we multiply the last column by -1 to get

$$\begin{pmatrix} 4 & 1 \\ 15 & 3 \end{pmatrix}.$$

The next step is to switch the first and second columns.

$$\begin{pmatrix} 1 & 4 \\ 3 & 15 \end{pmatrix}.$$

Now we use the entry in the first row and first column to eliminate all the other entries in the first row and first column. We take multiply the first row by -3 and add it to the second row.

$$\begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}.$$

Now we use the first column to eliminate the entry in the first row.

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

We put this back into the original 3×3 matrix, remembering to double every entry.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

(ii) We already checked that the gcd of the entries of A (the 1×1 minors) is 1. As the second row is even (that is, every entry in the second row is even) it is easy to see that every 2×2 minor is even. Thus the gcd is even.

In fact we can divide the second row by 2

$$\begin{pmatrix} -4 & -6 & 7 \\ 1 & 1 & 2 \\ 6 & 6 & 15 \end{pmatrix}$$

and we just need to show that the gcd of the 2×2 minors is 1 and that the absolute value of the determinant (3×3 minor) is 6. The top left 2×2 minor is 2. The bottom right 2×2 minor is 3. The gcd of these number is 1 and so the gcd of the 2×2 minors is one, as expected.
To compute the determinant, we first divide the third row by 3, to get

$$\begin{pmatrix} -4 & -6 & 7 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$$

The determinant is 2, as expected.
(iii) Each row and column operation may be represented by a 3×3 matrix. Just take the identity matrix and apply one of the elementary operations. For example if you pre-multiply A by one of the three matrices

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

you will either multiply the first row by -2, or you will swap the first and second rows, or you will take the first row, multiply it -5 and add it to the second row.
Similar considerations apply to the elementary column operations, except now this is represented by post-multiplication. Now to get from A to D we simply applied a bunch of row and column operations. This is encoded by a product of matrices Q and P (note that the order in which we apply the operations does not matter, as matrix multiplication is associative). As each elementary matrix is invertible and the product of invertible matrices is invertible, P and Q are both invertible.
4. We start with the first matrix.

$$A = \begin{pmatrix} 2x-1 & x & x-1 & 1 \\ x & 0 & 1 & 0 \\ 0 & 1 & x & x \\ 1 & x^2 & 0 & 2x-2 \end{pmatrix}.$$

The gcd of the entries is obviously one. Let us swap the first and third columns and then the first and second rows to get

$$\begin{pmatrix} x-1 & x & 2x-1 & 1 \\ 1 & 0 & x & 0 \\ x & 1 & 0 & x \\ 0 & x^2 & 1 & 2x-2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & x & 0 \\ x-1 & x & 2x-1 & 1 \\ x & 1 & 0 & x \\ 0 & x^2 & 1 & 2x-2 \end{pmatrix}.$$

Now let us use the top left entry to eliminate the entries in the first column. We multiply the first row by $x-1$ and subtract it from the second row and then multiply the first row by x and subtract it from the third row:

$$\begin{pmatrix} 1 & 0 & x & 0 \\ 0 & x & -x^2+3x-1 & 1 \\ x & 1 & 0 & x \\ 0 & x^2 & 1 & 2x-2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & x & 0 \\ 0 & x & -x^2+3x-1 & 1 \\ 0 & 1 & -x^2 & x \\ 0 & x^2 & 1 & 2x-2 \end{pmatrix}.$$

Now we use the first column to eliminate the entries in the first row and then we swap the second and third rows to get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & -x^2+3x-1 & 1 \\ 0 & 1 & -x^2 & x \\ 0 & x^2 & 1 & 2x-2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -x^2 & x \\ 0 & x & -x^2+3x-1 & 1 \\ 0 & x^2 & 1 & 2x-2 \end{pmatrix}.$$

Now we multiply the second row by $-x$ and add it to the third row and then by $-x^2$ and add it to the fourth row.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -x^2 & x \\ 0 & 0 & x^3-x^2+3x-1 & 1-x^2 \\ 0 & x^2 & 1 & 2x-2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -x^2 & x \\ 0 & 0 & x^3-x^2+3x-1 & 1-x^2 \\ 0 & 0 & x^4+1 & -x^3+2x-2 \end{pmatrix}.$$

Note that the gcd of the entries of the bottom right 2×2 submatrix is 1. Instead of continuing, to compute the last entry just take the determinant,

$$d(x) = x^5 - 2x^4 - 3x^3 + 9x^2 - 8x + 1.$$

It follows that the Smith normal form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d(x) \end{pmatrix}.$$

For the second matrix

$$B = \begin{pmatrix} x^2+2x & 0 & 0 & 0 \\ 0 & x^2+3x+2 & 0 & 0 \\ 0 & 0 & x^3+2x^2 & 0 \\ 0 & 0 & 0 & x^4+x^3 \end{pmatrix}$$

this matrix is almost already in Smith normal form. It is much better to use this fact rather than try elimination.
We have

$$\begin{aligned} x^2+2x &= x(x+2) \\ x^2+3x+2 &= (x+1)(x+2) \\ x^3+2x^2 &= x^2(x+2) \\ x^4+x^3 &= x^3(x+1). \end{aligned}$$

To find the gcd of the products, we just need to consider the largest powers of x, $x+1$ and $x+2$ which divides every product.
The gcd of these polynomials is 1. The gcd of pairwise products is $x(x+2)$. The gcd of triple products is $x^3(x+1)(x+2)^2$. The product is $x^6(x+1)^2(x+2)^3$. The ratios are

$$1 \qquad x(x+2) \qquad x^2(x+1)(x+2) \qquad \text{and} \qquad x^3(x+1)(x+2).$$

Thus the Smith normal form is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x(x+2) & 0 & 0 \\ 0 & 0 & x^2(x+1)(x+2) & 0 \\ 0 & 0 & 0 & x^3(x+1)(x+2) \end{pmatrix}.$$

5. G is isomorphic to

$$\frac{\mathbb{Z}^3}{K} \qquad \text{where} \qquad K = \langle (6,6,0), (10,0,10), (0,15,15) \rangle.$$

We want a linear map with image K. We write down the matrix whose columns are the generators for K (since the columns of a matrix span the image of the linear map):

$$A = \begin{pmatrix} 6 & 10 & 0 \\ 6 & 0 & 15 \\ 0 & 10 & 15 \end{pmatrix}.$$

This data is encoded by the group homomorphism (or $\mathbb{Z}$-linear map)

$$\mathbb{Z}^3 \longrightarrow \mathbb{Z}^3 \qquad \text{given by} \qquad (x,y,z) \longrightarrow (6x+10y, 6x+15z, 10y+15z),$$

since then the image is K.

We have to put A into Smith normal form. The easiest thing to do is probably to compute the gcd of the minors, since A is almost in Smith normal form. The entries of A have prime factors 2, 3 and 5. But one entry is odd, one entry, 10, is not divisible by 3 and one entry, 6, is not divisible by 5. Thus the gcd of the entries of A is one. Every 2×2 minor has at least one zero in it. So each 2×2 minor is a product of two entries of A. The minors, up to sign, are

$$150 \quad 90 \quad 60 \quad 150 \quad 90 \quad 60 \quad 150 \quad 90 \quad \text{and} \quad 60.$$

The gcd of the minors is then $2 \cdot 3 \cdot 5 = 30$. Finally the determinant is

$$-6 \cdot 10 \cdot 15 - 10 \cdot 6 \cdot 15 = -1800$$

Thus the Smith normal form is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 60 \end{pmatrix}.$$

The quotient group of the corresponding linear map is then

$$\mathbb{Z}_{30} \times \mathbb{Z}_{60}.$$

6. A is a 9×9 matrix, as the characteristic polynomial has degree 9. The entries on the main diagonal are the zeroes of the characteristic polynomial. Thus there are 6 minus ones and 3 twos.
As the minimal polynomial has $(x+1)^3$ as a factor it follows that there is a 3×3 (and no larger) Jordan block with -1 on the main diagonal.
As the minimal polynomial has $(x-2)^2$ as a factor it follows that there is a 2×2 (and no larger) Jordan block with 2 on the main diagonal.
Consider the Jordan blocks with eigenvalue -1. There is one of size 3. If the other Jordan blocks are of type $a_i \times a_i$, $a_1, a_2, \dots, a_k$ decreasing then we must have

$$\sum a_i = 6 - 3 = 3.$$

The three possible solutions are $k = 1$, $a_1 = 3$; $k = 2$, $a_1 = 2$ and $a_2 = 1$; $k = 3$, $a_1 = a_2 = a_3 = 1$.
Now consider the Jordan blocks with eigenvalue 2. There is one of size 2. The only possibility is that there is one more of size 1. There are thus three possibilities, the first, two 3×3 Jordan blocks with eigenvalue

-1

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

the second, one 3×3, one 2×2, one 1×1 Jordan block, eigenvalue -1

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and the third, one 3×3, and three 1×1 Jordan blocks, eigenvalue -1

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

7. Suppose that A is an invertible matrix. Note that A is similar to A' if and only if A' is invertible and A are conjugate in the general linear group.
Therefore we can describe the conjugacy classes by writing down a preferred member of the conjugacy class, using one of the canonical forms.
Since these fields are not algebraically closed, we use rational canonical form.
Note that invertible matrices don't have zero as an eigenvalue.

(i) The characteristic polynomial is a monic quadratic polynomial. The quadratic polynomials are

$$x^2 \qquad x^2+1=(x+1)^2 \qquad x^2+x=x(x+1) \qquad \text{and} \qquad x^2+x+1.$$

Recall that the last polynomial is irreducible. Since 0 is not an eigenvalue we can eliminate the first and third possibility. Thus the characteristic polynomial is either

$$(x+1)^2 \qquad \text{or} \qquad x^2+x+1.$$

The minimal polynomial divides the characteristic polynomial and has the same roots.
Thus the minimal polynomial is $x+1$, $(x+1)^2$, with characteristic polynomial $(x+1)^2$ or x^2+x+1, with the same characteristic polynomial.
The first possibility corresponds to the identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This corresponds to two copies of the companion matrix of $x+1$. The order is 1. If we have the second possibility then we have the companion matrix of x^2+1,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The order is 2. If we have the third possibility then we have the companion matrix of x^2+x+1

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The order is 3.
(ii) The characteristic polynomial is a monic cubic polynomial and zero is not a root. The cubic polynomials which don't have zero as a root are

$$(x+1)^3=x^3+x^2+x+1 \qquad x^3+x^2+1 \qquad \text{and} \qquad x^3+x+1$$

Recall that the last two polynomials are irreducible.
The minimal polynomial divides the characteristic polynomial and has the same roots.
Thus the minimal polynomial is $x+1$, $(x+1)^2$, or $(x+1)^3$, with characteristic polynomial $(x+1)^3$ or x^3+x^2+1 or x^3+x+1, with the same characteristic polynomial.

The first possibility corresponds to the identity matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This corresponds to three copies of the companion matrix of $x+1$. The order is 1. If we have the second possibility then we have one copy of the companion matrix of $x+1$ and one copy of the companion matrix of x^2+1,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The order is 2. If we have the third possibility then we have the companion matrix of x^3+x^2+x+1

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

The order is 4. If we have the fourth possibility then we have the companion matrix of x^3+x^2+1

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

The order is 7. If we have the fifth possibility then we have the companion matrix of x^3+x+1

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The order is 7.

Challenge Problems: (Just for fun)

(iii) First note that if A has determinant one and A' is similar to A then A' has determinant one. Suppose that

$$A' = BAB^{-1}.$$

Then B is an invertible matrix, so that the determinant b is non-zero. Then $b = 1$, ω or $1+\omega$. If $b = 1$ then let $c = 1$. If $b = \omega$ then let

$c = 1 + \omega$. We have

$$\begin{aligned} c^2 &= (1+\omega)^2 \\ &= 1^2 + \omega^2 \\ &= 1 + \omega + 1 \\ &= \omega \\ &= b. \end{aligned}$$

If $b = 1 + \omega$ then let $c = \omega$. We have

$$\begin{aligned} c^2 &= \omega^2 \\ &= 1 + \omega \\ &= b. \end{aligned}$$

Thus b always has a square root c. If

$$C = \frac{1}{c} B$$

then C is invertible,

$$\begin{aligned} \det C &= \frac{1}{c^2} \det B \\ &= \frac{b}{b} \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} CAC^{-1} &= \frac{c}{c} BAB^{-1} \\ &= A'. \end{aligned}$$

Thus A and A' are conjugate in

$$\mathrm{SL}_2(\mathbb{F}_4).$$

Thus we just need to find all possible rational canonical forms.
We start by determining all of the characteristic polynomials. Note that if we start with $A - xI_2$ and set $x = 0$ then we get A. Thus the determinant of A is the constant term of the characteristic polynomial. Thus the characteristic polynomial has constant term 1.
The characteristic polynomial is a monic quadratic polynomial. The quadratic polynomials with constant term one are

$$x^2+1 = (x+1)^2 \qquad x^2+x+1 \qquad x^2+\omega x+1 \qquad \text{and} \qquad x^2+(1+\omega)x+1.$$

By construction, ω is a root of the second polynomial. It is easy to check that $1 + \omega$ is the other root. It follows that the last two polynomials are irreducible.

The minimal polynomial divides the characteristic polynomial and it has the same roots. Thus the minimal polynomial is $x+1$ or it is equal to the characteristic polynomial.
The first possibility corresponds to the identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This corresponds to two copies of the companion matrix of $x+1$. The order is 1. If we have the second possibility then we have the companion matrix of x^2+1,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The order is 2. If we have the third possibility then we have the companion matrix of x^2+x+1

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The order is 3. If we have the fourth possibility then we have the companion matrix of $x^2+\omega x+1$

$$\begin{pmatrix} 0 & 1 \\ 1 & omega \end{pmatrix}.$$

The order is 5. If we have the fifth possibility then we have the companion matrix of $x^2+(1+\omega)x+1$

$$\begin{pmatrix} 0 & 1 \\ 1 & 1+omega \end{pmatrix}.$$

The order is 5.
8. (i) Clear, since

$$G = \{\,(r_1, r_2, \dots, r_n - 1., 0) \mid r_1, r_2, \dots, r_{n-1}\,\} \simeq R^{n-1}.$$

(ii) Induction on n.
(iii) Q is a submodule of R and so Q is an ideal of R. As R is a PID, $Q = \langle a \rangle$, for some $a \in R$. As f is surjective, $a = f(e)$, some e.
(iv) Suppose that $m \in M$. Then $f(m) \in Q$ and so $f(m) = ra$, some $r \in R$. Let $n = m - re$. Then

$$\begin{aligned} f(n) &= f(m - re) \\ &= f(m) - rf(e) \\ &= f(m) - ra \\ &= f(m) - f(m) \\ &= 0. \end{aligned}$$

Thus $n \in N$. It follows that we may find $r_1, r_2, \dots, r_l$ such that

$$n = \sum_{i \leq l} r_i f_i.$$

In this case

$$n = \sum_{i \leq l} r_i f_i + re.$$

Thus N is generated by $f_1, f_2, \dots, f_l, e$. Note that r is determined, so that $r_1, r_2, \dots, r_l$ are determined and so $f_1, f_2, \dots, f_l, e$ are free generators of M.

(v) Thus M is a free module of rank $l + 1 \leq n$.

9. Note that A is a zero of $x^2+1 \in \mathbb{R}[x]$. Thus the minimal polynomial $m_A(x)$ divides $x^2 + 1$. As $x^2 + 1$ is irreducible and monic in fact

$$m_A(x) = x^2 + 1.$$

We put A into canonical form. We choose the rational canonical form, as $\mathbb{R}$ is not algebraically closed. The companion matrix of $x^2 + 1$ is

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The rational canonical form for A consists of m blocks of these 2×2 matrices. Thus $n = 2m$ is even and the rational canonical form is unique.

There are now two ways to proceed. For the first we carefully choose a new basis for A, to put it into the desired form.

Or we can argue as follows. The matrix

$$B = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}$$

satisfies the equation

$$B^2 + I_n = 0.$$

As the rational canonical form is unique, there is only one conjugacy class for such matrices and so A is similar to B.

10. Define a sequence of functions

$$f_n \colon [0, 1] \longrightarrow \mathbb{R} \qquad \text{given by} \qquad f_n(x) = \cos 2\pi n x.$$

Then $f_1, f_2, \dots$ is an infinitely differentiable function.

Now consider the sequence of ideals

$$I_1 \subset I_2 \subset I_3 \subset \dots \qquad \text{given by} \qquad I_n = \langle f_1, f_2, \dots, f_n \rangle.$$

It is not hard to see that this sequence is strictly increasing. Thus R is not Noetherian.

11. Let J be a Jordan block of size n. Consider J^2. J has minimal polynomial x^n. If $n = 2m$ then J^2 has minimal polynomial $x^m m$ and if $n = 2m + 1$ then J^2 has minimal polynomial x^{m+1}. It is then not too hard to check that if $n = 2m$ is even then the Jordan canonical form for J^2 consists of two Jordan blocks of size m with the square of the eigenvalue for J and that if $n = 2m + 1$ is odd then the Jordan canonical form for J^2 is one Jordan block of size $m + 1$ and one of size m with the square of the eigenvalue for J.
(a) This is not possible. Consider the size of the Jordan blocks for A. There are none of size 1, since A^2 has no Jordan blocks of size 1. But then A^2 would have an even number of blocks.
(b) This is possible, if A has one Jordan block of size 8 and one of size 1.