MODEL ANSWERS TO THE EIGHTH HOMEWORK

1. (a) We have to define an R-linear map,

$$\phi\colon M \underset{R}{\otimes} N \longrightarrow N \underset{R}{\otimes} M.$$

By the universal property of $M \underset{R}{\otimes} N$, it suffices to exhibit a bilinear map

$$f\colon M \times N \longrightarrow N \underset{R}{\otimes} M$$

The composition of $u\colon N \times M \longrightarrow N \underset{R}{\otimes} M$ and the map

$$M \times N \longrightarrow N \times M \qquad \text{which sends} \qquad (m,n) \longrightarrow (n,m)$$

will obviously do. The inverse map is constructed similarly. The composition either way is easily seen to be the identity, either because it satisfies the universal property of the identity, or because it is the identity map on generators.
(b) One can prove this as above. Here is a much sneakier way to proceed. Note that

$$(M \times N) \times P \qquad \text{and} \qquad M \times (N \times P)$$

are in obvious bijection.
On the other hand, given either triple product, one can consider trilinear maps, that is, maps that are linear in all three variables. It is not hard to check that $(M \underset{R}{\otimes} N) \underset{R}{\otimes} P$ satisfies the corresponding universal propery. Similarly for $M \underset{R}{\otimes} (N \underset{R}{\otimes} P)$. Thus they are canonically isomorphic.
(c) We are going to show that M satisfies the properties of the tensor product. First we need to exhibit a bilinear map,

$$u\colon R \times M \longrightarrow M$$

The definition of u is almost forced, send (r,m) to rm. This is clearly a bilinear map. Now suppose we are given a bilinear map

$$f\colon R \times M \longrightarrow N.$$

Define

$$\phi\colon M \longrightarrow N$$

by sending m to $f(1, m)$. We check that the diagram,

$$\begin{tikzcd} R \times M \arrow[r, "f"] \arrow[d, "u"'] & N \\ M \arrow[ur, dotted, "\phi"] & \end{tikzcd}$$

commutes. Suppose that $(r, m) \in R \times M$. Then

$$\begin{aligned} \phi \circ u(r, m) &= \phi(rm) \\ &= f(1, rm) \\ &= rf(1, m) \\ &= f(r, m), \end{aligned}$$

where we applied bilinearity of f twice. Thus the diagram commutes. Finally we check that ϕ is R-linear. Suppose that m_1, $m_2 \in M$. Then

$$\begin{aligned} \phi(m_1 + m_2) &= f(1, m_1 + m_2) \\ &= f(1, m_1) + f(1, m_2) \\ &= \phi(m_1) + \phi(m_2). \end{aligned}$$

Now suppose that $r \in R$ and $m \in M$. Then

$$\begin{aligned} \phi(rm) &= f(1, rm) \\ &= rf(1, m) \\ &= r\phi(m). \end{aligned}$$

Thus ϕ is R-linear. Thus M satisfies all the properties of a tensor product and the result is clear.

(d) First we define a bilinear map

$$M \times (N \oplus P) \longrightarrow (M \underset{R}{\otimes} N) \oplus (M \underset{R}{\otimes} P),$$

by sending $(m, (n, p))$ to $(m \otimes n, m \otimes p)$. It is easy to check that this is bilinear. This gives us a map one way. To get a map the other way, it suffices, by definition of the direct sum and then of the tensor product and by symmetry, to exhibit a bilinear map

$$M \times N \longrightarrow M \underset{R}{\otimes} (N \oplus P).$$

For this send (m, n) to $m \otimes (n, 0)$. Again it is clear that this map is bilinear and that the induced R-linear maps are inverse to each other.

(e) As $F \simeq R^n$, this follows immediately from (c) and (d), by induction on n.

2. Let d be the gcd of m and n. I claim that

$$\mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n \simeq \mathbb{Z}_d.$$

The proof proceeds in two steps. First observe that

$$\begin{aligned} m(1 \otimes 1) &= m \otimes 1 \\ &= 0 \otimes 1 \\ &= 0. \end{aligned}$$

Similarly $n(1 \otimes 1) = 0$. As $\mathbb{Z}$ is a PID, we may find r and s such that

$$d = rm + sn.$$

Thus

$$\begin{aligned} d(1 \otimes 1) &= (rm + sn)1 \otimes 1 \\ &= r(m(1 \otimes 1) + s(n(1 \otimes 1)) \\ &= 0. \end{aligned}$$

Thus $\mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n$ is surely isomorphic to a subgroup of $\mathbb{Z}_d$. It remains to check that no smaller multiple of $1 \otimes 1$ is zero. The best way to prove this is to use the universal property. Let

$$f\colon \mathbb{Z}_m \times \mathbb{Z}_n \longrightarrow \mathbb{Z}_d$$

be the map that sends (a, b) to ab. As d divides both m and n, this map is indeed well-defined. On the other hand it is clearly bilinear. By the universal property, it induces an R-linear map

$$\phi\colon \mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n \longrightarrow \mathbb{Z}_d.$$

This map sends $1 \otimes 1$ to $f(1, 1)$, that is, 1. Hence if $k(1 \otimes 1) = 0$, then k is zero in $\mathbb{Z}_d$ and so d divides k. The result follows.

3. We first prove that $M \underset{R}{\otimes} N$ is finitely generated. Suppose that $x_1, x_2, \dots, x_m$ and $y_1, y_2, \dots, y_n$ are generators of M and N. Then I claim that $x_i \otimes y_j$ are generators of $M \underset{R}{\otimes} N$. Indeed this is generated by elements of the form $m \otimes n$, and so it is enough to observe that if

$$m = \sum r_i x_i \qquad \text{and} \qquad n = \sum s_i n_i,$$

then

$$m \otimes n = \sum (r_i s_j) x_i \otimes y_j,$$

where of course we use bilinearity to distribute the sum.
If R is Noetherian then $M \underset{R}{\otimes} N$ is a finitely generated module over a Noetherian ring so that $M \underset{R}{\otimes} N$ is Noetherian.

Challenge Problems: (Just for fun)

4. Any finitely generated abelian group is a direct sum of cyclic groups. As the tensor product distributes over the direct sum, by (1) (d), it is enough to determine the tensor product of two cyclic groups. Since the tensor product is commutative, we have to calculate three products:

$$\mathbb{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Z} \qquad \mathbb{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n \qquad \text{and} \qquad \mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n.$$

If we apply 1 (c) to the first two products and 2 to the last we get

$$\mathbb{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Z} \simeq \mathbb{Z} \qquad \mathbb{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n \simeq \mathbb{Z}_n \qquad \text{and} \qquad \mathbb{Z}_m \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_n \simeq \mathbb{Z}_d.$$

5. Consider

$$\frac{a}{b} \otimes \frac{c}{d},$$

where a, b, c and d are all integers, and $bd \neq 0$. We may suppose that $d > 0$. In this case

$$\begin{aligned} 0 &= \frac{a}{bd} \otimes 0 \\ &= \frac{a}{bd} \otimes c \\ &= d(\frac{a}{bd} \otimes \frac{c}{d}) \\ &= \frac{a}{b} \otimes \frac{c}{d}. \end{aligned}$$

Thus

$$\mathbb{Q}/\mathbb{Z} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}/\mathbb{Z} \simeq 0.$$

6. First observe that the direct sum M^n is Noetherian, by induction on n, applied to the standard short exact sequence:

$$0 \longrightarrow M^{n-1} \longrightarrow M^n \longrightarrow M \longrightarrow 0.$$

It follows that

$$M \underset{R}{\otimes} R^n \simeq (M \underset{R}{\otimes} R)^n \simeq M^n,$$

is Noetherian. By assumption there is a surjective R-linear map

$$R^n \longrightarrow N,$$

for some n. If we tensor this by M we get a surjective R-linear map (it is a general fact that surjective maps remain surjective after tensoring)

$$M \underset{R}{\otimes} R^n \longrightarrow M \underset{R}{\otimes} N.$$

Thus $M \underset{R}{\otimes} N$ is a quotient of a Noetherian R-module, so that it is Noetherian.

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