MODEL ANSWERS TO THE SEVENTH HOMEWORK

1. (i) Probably the easiest example is to take the zero ideal in $\mathbb{Z}$. This is prime, as $\mathbb{Z}$ is an integral domain, but it is not maximal as the quotient, $\mathbb{Z}$, is not a field.
(ii) Take the example given in (iii).
(iii) By Gauss' Lemma, $\mathbb{Z}[x]$ is a UFD. On the other hand I claim the ideal $I = \langle 2, x\rangle$ is not principal. Indeed suppose it was, so that $\langle 2, x\rangle = \langle f(x)\rangle$. As $2 \in I$ it follows that $f(x)$ divides 2. Up to associates, it would then follow that $f(x) = 1$ or 2. By the same token, $f(x)$ must divide x as well, and so $f(x) = 1$. But this is a contradiction, as $1 \notin I$.
2. We have to check that N is non-empty and closed under addition and scalar multiplication.
N is clearly non-empty as N_1 is non-empty.
Suppose that a and $b \in N$. Then there are indices i and j such that $a \in N_i$ and $b \in N_j$. Suppose that k is the maximum of i and j. Then $a \in N_k$ and $b \in N_k$. As N_k is closed under addition then $a + b \in N_k$. But then $a + b \in N$. Thus N is closed under addition.
Now suppose that $a \in N$ and $r \in R$. Then $a \in N_i$ some i. As N_i is closed under scalar multiplication, $ra \in N_i$. But then $ra \in N$ and so N is closed under scalar multiplication.

Challenge Problems: (Just for fun)

3. (a) Suppose that $n_1, n_2, \dots, n_k$ generate N so that $N = \langle n_1, n_2, \dots, n_k\rangle$. Let $p_1, p_2, \dots, p_k$ be the images of $n_1, n_2, \dots, n_k$. Then $p_1, p_2, \dots, p_k$ generates the image of N, which is P. Thus P is finitely generated.
Now suppose that P is finitely generated. Let $m_1, m_2, \dots, m_a$ and $p_1, p_2, \dots, p_b$ be generators of M and P. Let $n_1, n_2, \dots, n_a$ be the images of $m_1, m_2, \dots, m_a$. As $N \longrightarrow P$ is surjective we may pick $n_{a+i} \in N$ mapping to p_i. We claim that $n_1, n_2, \dots, n_c$ generates N, where $c = a + b$.
Suppose that we start with $n \in N$. Let p be the image in P. Then we may find $r_i \in R$, $a < i < b$, such that

$$p = \sum_i r_{a+i} p_i.$$

Let

$$q = \sum_{i>a} r_i n_i,$$

so that q maps to p. Let

$$z = n - q.$$

Then z is sent to zero, so that we may find $m \in M$ mapping to z. Therefore we may find $r_1, r_2, \dots, r_a$ such that $m = \sum_i r_i m_i$. In this case

$$z = \sum_i r_i n_i.$$

It follows that

$$\begin{aligned} n &= z + q \\ &= \sum_{i \leq a} r_i n_i + \sum_{a < i \leq c} r_i n_i \\ &= \sum_i r_i n_i. \end{aligned}$$

Thus $N = \langle n_1, n_2, \dots, n_c \rangle$ so that N is finitely generated.
(b) Let us give names to the two maps in the short exact sequence, i and π.
Pick free generators X of P. Pick $Y \subset N$ such that the map $Y \longrightarrow X$ induced by π is a bijection. In other words, for every $x \in X$, pick $y \in Y$ mapping to x, so that $\pi(y) = x$. Let

$$f \colon X \longrightarrow Y$$

be the inverse function. By the universal property of P we may find an R-linear map

$$\phi \colon P \longrightarrow N$$

which extends f. Note that the composition of f and $Y \longrightarrow X$ is the identity, so that the composition of ϕ and $\pi \colon N \longrightarrow P$ is also the identity (since the identity $P \longrightarrow P$ is an R-linear map that extends the identity $X \longrightarrow X$ and there is only one map that works). The map ϕ is called a **splitting of the exact sequence**.
We show that the existence of a splitting implies that the short exact sequence splits (in fact the sequence splits if and only if there is a splitting).
So now we have two R-linear maps, $i \colon M \longrightarrow N$ and $\phi \colon P \longrightarrow N$. This induces an R-linear map

$$\psi \colon M \oplus P \longrightarrow N$$

by the universal property of the direct sum. In fact

$$\psi(m, p) = i(m) + \phi(p).$$

We have

$$\begin{aligned}(\pi \circ \psi)(m,p) &= \pi(\psi(m,p)) \\ &= \pi(i(m) + \phi(p)) \\ &= \pi(i(m)) + \pi(\phi(p)) \\ &= 0 + p \\ &= p.\end{aligned}$$

In particular if $(m,p) \in \operatorname{Ker}(\psi)$ then $p = 0$, so that $i(m) = 0$, so that $m = 0$.

Hence ψ is injective. Now suppose that $n \in N$. Let $p = \pi(n)$ and let $n' = \phi(p)$.

Consider $n - n'$. We have

$$\pi(n - n') = 0,$$

so that we may find $m \in M$ such that $i(m) = n - n'$. It follows that

$$\psi(m,p) = i(m) + \phi(p) = n.$$

Hence ψ is surjective. Thus ψ is an isomorphism and so the short exact sequence does indeed split.

Note that one can also show that the sequence splits if and only if there an R-linear map $\xi\colon N \longrightarrow M$ such that the composition of $i\colon M \longrightarrow N$ and $\xi\colon N \longrightarrow M$ is the identity.

(c) Consider the short exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}_2 \longrightarrow 0,$$

It is clear that $\mathbb{Z}$ is not isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$, so that this sequence does not split.

4. (a) We first show that n is injective. This involves "diagram chasing".

Pick $c \in C$ in the kernel of n. Let $d \in D$ be the image of c. Then d maps to zero in D'. Indeed let $d' \in D'$ be the image of d. If c' is the image of c in C' then c' maps to d', as the square containing C, D, C' and D' commutes.

But $c' = 0$ is zero by hypothesis and so $d' = 0$. As p is an isomorphism it is surely injective. Thus d must be zero. As the top row is exact it follows that we may find $b \in B$ mapping to c. Let b' be the image of b. Then b' maps to $c' = 0$, as the diagram commutes (we just need commutativity of the appropriate square).

As the bottom row is exact we must be able to find $a' \in A'$ mapping to b'. As l is surjective it follows that we may find $a \in A$ mapping to a'. If βB is the image of a then β is sent to b', as the first square commutes. But b is also sent to b'. As m is injective it follows that $\beta = b$.

As the top row is exact it follows that b is sent to $0 \in C$. But the image of b is c, so that $c = 0$. Hence n is injective.

Careful examination of the proof tells us that if we have a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "l"'] & B \arrow[r] \arrow[d, "m"'] & C \arrow[r] \arrow[d, "n"'] & D \arrow[d, "p"'] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D'
\end{tikzcd}$$

of R-modules with exact rows such that if m and p are injective and l is surjective, then n is injective. This is one of the four lemmas.

Now we turn to surjectivity. As the proof is similar to injectivity we give less details.

We start with $c' \in C'$. Let $d' \in D'$ be the image of c'. Then we may find $d \in D$ mapping to d' as p is surjective. Let $e \in E$ be the image of d and let $e' \in E'$ be the image of e. Then e' is also the image of d'. As d' is the image of c' it follows that $e' = 0$. As q is injective it follows that $e = 0$.

But then d must be the image of $c \in C$. Let $c'' \in C'$ be the image of c. Then c'' is sent to d' as c is sent to d. It follows the difference $\gamma' = c' - c''$ is sent to zero. But then γ' is the image of $\beta' \in B'$. As m is surjective we may find $\beta \in B$ mapping to $\beta' \in B'$. Let $\gamma \in C$ be the image of β. Then γ maps to γ'.

Consider $c + \gamma \in C$. This maps to

$$c'' + \gamma' = c'' + (c' - c'') = c'.$$

Thus n is surjective.

Once again, looking carefully at what we actually use in the proof of surjectivity gives us the other four lemma:

If we have a commutative

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "m"'] & C \arrow[r] \arrow[d, "n"'] & D \arrow[r] \arrow[d, "p"'] & E \arrow[d, "q"'] \\
B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

of R-modules with exact rows such that if m and p are surjective and q is injective, then n is surjective.

(b) By assumption, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, phantom, "="] & M_i \arrow[r] \arrow[d, phantom, "="] & N_i \arrow[r] \arrow[d, "n"'] & P_i \arrow[r] \arrow[d, phantom, "="] & 0 \arrow[d, phantom, "="] \\
0 \arrow[r] & M_j \arrow[r] & N_j \arrow[r] & P_j \arrow[r] & 0
\end{tikzcd}$$

with exact rows. The equals signs indicates that we have the same object. The four equals signs gives rise to four downward R-linear maps, which are all isomorphisms.
In particular the two extreme vertical maps are surjective and injective.
It follows that the middle map is an isomorphism.
As this map is an inclusion map we must have $N_i = N_j$.