5. Prime and Maximal Ideals

Let R be a ring and let I be an ideal of R, where $I \neq R$. Consider the quotient ring R/I. Two very natural questions arise:

(1) When is R/I a domain?
(2) When is R/I a field?

Definition-Lemma 5.1. *Let R be a ring and let I be an ideal of R. We say that I is **prime** if $I \neq R$ and whenever $ab \in I$ then either $a \in I$ or $b \in I$.*

R/I is a domain if and only if I is prime.

Proof. Suppose that I is prime. Let x and y be two elements of R/I. Then there are elements a and b of R such that $x = a+I$ and $y = b+I$. Suppose that $xy = 0$, but that $x \neq 0$, that is, suppose that $a \notin I$.

$$\begin{aligned} xy &= (a+I)(b+I) \\ &= ab + I \\ &= 0. \end{aligned}$$

But then $ab \in I$ and as I is prime, $b \in I$. But then $y = b + I = I = 0$. Thus R/I is a domain.

Now suppose that R/I is a domain. Let a and b be two elements of R such that $ab \in I$ and suppose that $a \notin I$. Let $x = a + I$, $y = b + I$. Then $xy = ab + I = 0$. As $x \neq 0$, and R/I is a domain, $y = 0$. But then $b \in I$ and so I is prime. □

Example 5.2. *Let $R = \mathbb{Z}$. Then every ideal in R has the form $\langle n \rangle = n\mathbb{Z}$. It is not hard to see that I is prime if and only if n is prime.*

Definition 5.3. *Let R be an integral domain and let a be a non-zero element of R. We say that a is **prime**, if $\langle a \rangle$ is a prime ideal.*

Note that the condition that $\langle a \rangle$ is not the whole of R is equivalent to requiring that a is not invertible.

Definition-Lemma 5.4. *Let R be a ring. Then there is a unique ring homomorphism $\phi\colon \mathbb{Z} \longrightarrow R$.*

*We say that the **characteristic** of R is n if the order of the image of ϕ is finite, equal to n; otherwise the characteristic is 0.*

Let R be a domain of finite characteristic. Then the characteristic is prime.

Proof. Let $\phi\colon \mathbb{Z} \longrightarrow R$ be a ring homomorphism. Then $\phi(1) = 1$. Note that $\mathbb{Z}$ is a cyclic group under addition. Thus there is a unique map that sends 1 to 1 and is a group homomorphism. Thus ϕ is certainly unique and it is not hard to check that in fact ϕ is a ring homomorphism.

Now suppose that R is a domain. Then the image of ϕ is a domain. In particular the kernel I of ϕ is a prime ideal. Suppose that $I = \langle p \rangle$. Then the image of ϕ is isomorphic to R/I, that is the integers modulo p, and so the characteristic is equal to p. □

Another, obviously equivalent, way to define the characteristic n is to take the minimum non-zero positive integer such that $n1 = 0$.

Example 5.5. *The characteristic of $\mathbb{Q}$ is zero. Indeed the natural map $\mathbb{Z} \longrightarrow \mathbb{Q}$ is an inclusion. Thus every field that contains $\mathbb{Q}$ has characteristic zero. On the other hand $\mathbb{Z}_p$ is a field of characteristic p.*

Definition 5.6. *Let I be an ideal. We say that I is* ***maximal*** *if for every ideal J, such that $I \subset J$, either $J = I$ or $J = R$.*

Proposition 5.7. *Let R be a commutative ring.*

Then R is a field if and only if the only ideals are $\{0\}$ and R.

Proof. We have already seen that if R is a field, then R contains no non-trivial ideals.

Now suppose that R contains no non-trivial ideals and let $a \in R$. Suppose that $a \neq 0$ and let $I = \langle a \rangle$. Then $I \neq \{0\}$. Thus $I = R$. But then $1 \in I$ and so $1 = ba$. Thus a is a invertible and as a was arbitrary, R is a field. □

Theorem 5.8. *Let R be a commutative ring.*

Then R/M is a field if and only if M is a maximal ideal.

Proof. Note that there is an obvious correspondence between the ideals of R/M and ideals of R that contain M. The result follows immediately from (5.7). □

Corollary 5.9. *Let R be a commutative ring.*

Then every maximal ideal is prime.

Proof. Clear as every field is an integral domain. □

Example 5.10. *Let $R = \mathbb{Z}$ and let p be a prime. Then $I = \langle p \rangle$ is not only prime, but it is in fact maximal. Indeed the quotient is $\mathbb{Z}_p$.*

Example 5.11. *Let X be a set and let R be a commutative ring and let F be the set of all functions from X to R.*

Let $x \in X$ be a point of X and let I be the ideal of all functions vanishing at x. Then F/I is isomorphic to R.

Thus I is prime if and only if R is an integral domain and I is maximal if and only if R is a field. For example, take $X = [0, 1]$ and $R = \mathbb{R}$. In this case it turns out that every maximal ideal is of the same form (that is, the set of functions vanishing at a point).

Example 5.12. *Let R be the ring of Gaussian integers and let I be the ideal of all Gaussian integers $a+bi$ where both a and b are divisible by* 3*.*

I claim that I is maximal.

Indeed it is not hard to see that R/I is finite. As every finite integral domain is a field, in fact it suffices to prove that I is prime. Suppose that $(a+bi)(c+di) \in I$. As

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i,$$

we have

$$3|(ac-bd) \qquad \text{and} \qquad 3|(ad+bc).$$

Suppose that $a+bi \notin I$. Adding and subtracting the two results above we have

$$3|(a+b)c-(b-a)d \qquad \text{and} \qquad 3|(a+b)d+(b-a)c.$$

Now either 3 divides a and it does not divide b, or vice-versa, or the same is true, with $a+b$ replacing a and $a-b$ replacing b, as can be seen by an easy case-by-case analysis. Suppose that 3 divides a whilst 3 does not divide b. Then $3|bd$ and so $3|d$ as 3 is prime. Similarly $3|c$. Thus we are done in this case. Similar analyses pertain in the other cases.

Thus I is prime. It turns out that R/I is a field with nine elements.

Example 5.13. *Now suppose that we replace* 3 *by* 5 *and look at the resulting ideal J. I claim that J is not maximal.*

Indeed consider $x = 2+i$ and $y = 2-i$. Then

$$xy = (2+i)(2-i) = 4+1 = 5,$$

so that $xy \in J$, whilst neither x nor y are in J.

Thus J is not even prime.