4. Field of fractions

The rational numbers $\mathbb{Q}$ are constructed from the integers $\mathbb{Z}$ by adding inverses. In fact a rational number is of the form a/b, where a and b are integers. Note that a rational number does not have a unique representative in this way. In fact

$$\frac{a}{b} = \frac{ka}{kb}.$$

So really a rational number is an equivalence class of pairs $[a, b]$, where two such pairs $[a, b]$ and $[c, d]$ are equivalent if and only if $ad = bc$.

Now given an arbitrary integral domain R, we can perform the same operation.

Definition-Lemma 4.1. *Let R be any integral domain. Let N be the subset of $R \times R$ such that the second coordinate is non-zero.*

Define an equivalence relation $\sim$ on N as follows.

$$(a, b) \sim (c, d) \qquad \text{if and only if} \qquad ad = bc.$$

Proof. We have to check three things, reflexivity, symmetry and transitivity.

Suppose that $(a, b) \in N$. Then

$$a \cdot b = a \cdot b$$

so that $(a, b) \sim (a, b)$. Hence $\sim$ is reflexive.

Now suppose that (a, b), $(c, d) \in N$ and that $(a, b) \sim (c, d)$. Then $ad = bc$. But then $cb = da$, as R is commutative and so $(c, d) \sim (a, b)$. Hence $\sim$ is symmetric.

Finally suppose that (a, b), (c, d) and $(e, f) \in R$ and that $(a, b) \sim (c, d)$, $(c, d) \sim (e, f)$. Then $ad = bc$ and $cf = de$. Then

$$\begin{aligned}(af)d &= (ad)f \\ &= (bc)f \\ &= b(cf) \\ &= (be)d.\end{aligned}$$

As $(c, d) \in N$, we have $d \neq 0$. Cancelling d, we get $af = be$. Thus $(a, b) \sim (e, f)$. Hence $\sim$ is transitive. □

Definition-Lemma 4.2. *The **field of fractions of** R, denoted F is the set of equivalence classes, under the equivalence relation defined above. Given two elements $[a, b]$ and $[c, d]$ define*

$$[a, b] + [c, d] = [ad + bc, bd] \qquad \text{and} \qquad [a, b] \cdot [c, d] = [ac, bd].$$

With these rules of addition and multiplication F becomes a field. Moreover there is a natural injective ring homomorphism

$$\phi\colon R \longrightarrow F,$$

so that we may identify R as a subring of F. In fact ϕ is universal amongs all such injective ring homomorphisms whose targets are fields.

Proof. First we have to check that this rule of addition and multiplication is well-defined. Suppose that $[a, b] = [a', b']$ and $[c, d] = [c', d']$. By commutativity and an obvious induction (involving at most two steps, the only real advantage of which is to simplify the notation) we may assume $c = c'$ and $d = d'$. As $[a, b] = [a', b']$ we have $ab' = a'b$. Thus

$$\begin{aligned}(a'd + b'c)(bd) &= (a'bd + bb'c)d \\ &= (ab'd + bb'c)d \\ &= (ad + bc)(b'd).\end{aligned}$$

Thus $[a'd + b'c, b'd] = [ad + bc, bd]$. Thus the given rule of addition is well-defined. It can be shown similarly (and in fact more easily) that the given rule for multiplication is also well-defined.

We leave it is an exercise for the reader to check that F is a ring under addition and that multiplication is associative. For example, note that $[0, 1]$ plays the role of 0 and $[1, 1]$ plays the role of 1.

Given an element $[a, b]$ in F, where $a \neq 0$, then it is easy to see that $[b, a]$ is the inverse of $[a, b]$. It follows that F is a field.

Define a map

$$\phi\colon R \longrightarrow F,$$

by the rule

$$\phi(a) = [a, 1].$$

Again it is easy to check that ϕ is indeed an injective ring homomorphism and that it satisfies the given universal property. □

Example 4.3. *If we take $R = \mathbb{Z}$, then of course the field of fractions is isomorphic to $\mathbb{Q}$.*

If R is the ring of Gaussian integers, then F is a copy of the subfield of $\mathbb{C}$ consisting of all complex numbers of the form $a + bi$ where now a and b are elements of $\mathbb{Q}$.

If $R = K[x]$, where K is a field, then the field of fractions is denoted $K(x)$. It consists of all rational functions, that is, all quotients

$$\frac{f(x)}{g(x)},$$

where f and g are polynomials with coefficients in K.