12. Tensor Product

We want to introduce a new, and somewhat subtle, operation on modules, somewhat akin to the direct sum. For operations such as the direct sum, quotient and so on, whilst by far the best method for understanding these operations is to use the accompanying universal property, it is true that one can get away with understanding these operations independently of understanding the universal property. On the other hand, it is almost inconceivable that one could really understand the tensor product without coming to terms with its universal property.

Definition 12.1. *Let M, N and P be three R-modules. We say that a map*

$$f\colon M \times N \longrightarrow P$$

*is **bilinear** if it is linear in each factor. That is, we have*

$$\begin{aligned} f(m_1 + m_2, n) &= f(m_1, n) + f(m_2, n) \qquad & f(rm, n) &= rf(m, n) \\ f(m, n_1 + n_2) &= f(m, n_1) + f(m, n_2) \qquad & f(m, rn) &= rf(m, n). \end{aligned}$$

Definition 12.2. *Let M and N be two R-modules. The **tensor product** of M and N, denoted*

$$M \underset{R}{\otimes} N$$

is an R-module, together with a bilinear map

$$u\colon M \times N \longrightarrow M \underset{R}{\otimes} N$$

which has the following universal property: Suppose that P is any R-module and let

$$f\colon M \times N \longrightarrow P$$

be a bilinear map. Then there is a unique induced module homomorphism

$$\phi\colon M \underset{R}{\otimes} N \longrightarrow P$$

such that the following diagram commutes,

$$\begin{tikzcd} M \times N \arrow[r, "f"] \arrow[d, "u"'] & P. \\ M \underset{R}{\otimes} N \arrow[ur, dashed, "\phi"'] & \end{tikzcd}$$

In other words, the tensor product is universal amongst all bilinear maps, in the sense that it replaces a map that is bilinear (namely f), by a map that is R-linear (namely ϕ). Note then, that using the standard arguments, the tensor product is unique, up to unique isomorphism. The first thing to prove is that the tensor product exists. The point to realise about the construction below, is that even though in principle one constructs the tensor product explicitly, in fact the construction offers very little help in computing the tensor product in an explicit example.

Lemma 12.3. *Let M and N be two R-modules.*

Then the tensor product of M and N exists.

Proof. Let F be the free R-module generated by all elements of $M \times N$. Thus elements of F are formal linear combinations,

$$r_1(m_1, n_1) + r_2(m_2, n_2) + \cdots + r_k(m_k, n_k),$$

where $r_1, r_2, \dots, r_k \in R$, $m_1, m_2, \dots, m_k \in M$ and $n_1, n_2, \dots, n_k \in N$. We are going to define a submodule G of F, by giving generators of G. Suppose that m_1 and m_2 are in M and that $n \in N$. Then

$$(m_1 + m_2, n) - (m_1, n) - (m_2, n)$$

is one of the generators of G. Similarly if $r \in R$ and $(m, n) \in M \times N$, then

$$r(m, n) - (rm, n)$$

is a generator of G. Similarly, if $m \in M$, n, n_1, $n_2 \in N$ and $r \in R$ then

$$(m, n_1 + n_2) - (m, n_1) - (m, n_2) \qquad \text{and} \qquad r(m, n) - (m, rn),$$

are also generators of G.

Define T to be the quotient of F by G, $T = F/G$. I claim that T is the tensor product of M and N. First define a map

$$u \colon M \times N \longrightarrow T$$

in an obvious way. u should be the composition of the natural inclusion $M \times N \longrightarrow F$ and the quotient map $F \longrightarrow T$. We need to check that u is bilinear and universal amongst all such bilinear maps. The important point is to check that u satisfies the universal property of the tensor product.

First we check bilinearity of u. We have to check four things. We check only one, and leave the rest to the reader. Suppose that m_1, $m_2 \in M$ and $n \in N$. As

$$(m_1 + m_2, n) - (m_1, n) - (m_2, n) \in G$$

it follows that

$$u(m_1 + m_2, n) = u(m_1, n) + u(m_2, n).$$

Now we check that u satisfies the universal property. So suppose that we are given a bilinear map

$$f\colon M \times N \longrightarrow P$$

where P is an R-module. By the universal property of a free module, f induces an R-module homomorphism $\psi\colon F \longrightarrow P$ that extends f. As f is bilinear, it follows that every generator of G is in the kernel of ψ, so that the kernel of ψ contains G. But then by the universal property of a quotient, there is a unique R-module homomorphism

$$\phi\colon T \longrightarrow P. \qquad \square$$

It is useful to introduce some notation. We let $m \otimes n$ denote the image of (m, n) under the universal map. It follows by the construction of (12.3) that every element of $M \underset{R}{\otimes} N$ is a linear combination of these basic elements,

$$\sum r_i(m_i \otimes n_i).$$

In fact this also follows from the universal property, since the smallest R-module that contains the image of $M \times N$ obviously satisfies the same universal property as the tensor product.

Example 12.4.

$$\mathbb{Z}_2 \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_3 \simeq 0.$$

There are several ways to see this. First note that $0 \otimes n = 0$, for any $n \in N$. Indeed,

$$\begin{aligned} 0 \otimes n &= (0 + 0) \otimes n \\ &= 0 \otimes n + 0 \otimes n, \end{aligned}$$

so that cancelling, $0 \otimes n = 0$.

Note also that

$$(ab)(1 \otimes 1) = a \otimes b,$$

so that every element of $\mathbb{Z}_2 \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_3$ is a multiple of

$$1 \otimes 1.$$

Now

$$\begin{aligned} 2(1 \otimes 1) &= 1 \otimes 1 + 1 \otimes 1 \\ &= (1 + 1) \otimes 1 \\ &= 0 \otimes 1 \\ &= 0. \end{aligned}$$

It follows then that $2(1 \otimes 1) = 0$. On the other hand

$$\begin{aligned} 3(1 \otimes 1) &= 1 \otimes 3 \\ &= 1 \otimes 0 \\ &= 0. \end{aligned}$$

Thus $3(1 \otimes 1) = 0$. Subtracting, we get $1 \otimes 1 = 0$. The result follows.

Another way to see this result, is to show that there are no non-trivial bilinear maps

$$f \colon \mathbb{Z}_2 \times \mathbb{Z}_3 \longrightarrow G,$$

for any abelian group G. Note that

$$f(a, b) = abf(1, 1),$$

so that it suffices to prove that $f(1, 1) = 0$, regardless of f and G.

Note that

$$\begin{aligned} f(0, b) &= f(0 + 0, b) \\ &= f(0, b) + f(0, b). \end{aligned}$$

Cancelling we have $f(0, b) = 0$, regardless of b. Similarly $f(a, 0) = 0$ for all a. On the other hand

$$\begin{aligned} f(1, 1) &= f(3, 1) \\ &= 3f(1, 1) \\ &= f(1, 3) \\ &= f(1, 0) \\ &= 0. \end{aligned}$$

Example 12.5. *Now suppose we look at* $\mathbb{Z}_4 \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_6$.

As before, every element of $\mathbb{Z}_4 \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_6$ is a multiple of $1 \otimes 1$. Thus we have a cyclic group and we just need to determine the order.

Note that

$$4(1 \otimes 1) = 4 \otimes 1 = 0 \times 0 = 0 \qquad \text{and} \qquad 6(1 \otimes 1) = (1 \otimes 6) = 0 \otimes 0 = 0.$$

Subtracting, we see that $2(1 \otimes 1) = 0$. Thus we have a cyclic group of order at most two.

In fact one can also see this using bilinear maps. If

$$f \colon \mathbb{Z}_4 \times \mathbb{Z}_6 \longrightarrow G,$$

is bilinear then

$$4f(1, 1) = f(4, 1) = f(0, 1) = 0 \qquad \text{and} \qquad 6f(1, 1) = f(1, 6) = f(1, 0) = 0.$$

Thus $2f(1, 1) = 0$, so that we have a cyclic group of order at most two.

I claim that $\mathbb{Z}_4 \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_6$ is isomorphic to a cyclic group of order two. To finish, we need to exhibit a non-trivial bilinear map.

Define a map

$$u \colon \mathbb{Z}_4 \times \mathbb{Z}_6 \longrightarrow \mathbb{Z}_2$$

by sending (a, b) to ab. Note that this map is well-defined, in the sense that if we picked different representatives for a and b we would still get the same answer, modulo 2. Note that $(1, 1)$ is sent to 1, so that this map is non-trivial. This establishes the claim.

In retrospect, we ought to be able to show that u is the universal bilinear map of the tensor product. Suppose we are given

$$f \colon \mathbb{Z}_4 \times \mathbb{Z}_6 \longrightarrow G$$

a bilinear map, where G is any abelian group. Define a map

$$\phi \colon \mathbb{Z}_2 \longrightarrow G$$

by sending 1 to $f(1, 1)$. We have to prove that ϕ is $\mathbb{Z}$-linear. It suffices to check that $(2, 2)$ is sent to zero in G. This is checked just the same as before.

This proves that the tensor product is a subgroup of $\mathbb{Z}_2$. As u is non-trivial, we are done.

It is also interesting to figure out what happens for vector spaces. Suppose that V and W are two vector spaces over a field F, of dimensions m and n. Then $V \otimes W$ is a vector space of dimension mn (cf homework 8). If m and n are finite and $e_1, e_2, \dots, e_m$ and $f_1, f_2, \dots, f_n$ are bases for V and W, then $e_i \otimes f_j$, $1 \leq i \leq m$ and $1 \leq j \leq n$ forms a basis for $V \otimes W$; indeed they certainly span and there are mn vectors of this form. Note then that the general element of $V \otimes W$ is of the form

$$\sum_{i,j} a_{ij}(e_i \otimes f_j).$$

In particular, most elements of $V \otimes W$ are not of the form $v \otimes w$.

The tensor product allows for some very nice constructions.

Definition-Lemma 12.6. *Let $\phi \colon R \longrightarrow S$ be a ring homomorphism and let M be an R-module. Considering S as a module over itself, we can consider S as an R-module, via the map ϕ. Then the R-module $M \underset{R}{\otimes} S$ is naturally an S-module, by* ***extension of scalars****.*

Proof. As $M \underset{R}{\otimes} S$ is an R-module, it is certainly an abelian group. It suffices then to construct a scalar multiplication

$$S \times M \underset{R}{\otimes} S \longrightarrow M \underset{R}{\otimes} S.$$

Fix $s \in S$. Then we have to construct a map

$$M \underset{R}{\otimes} S \longrightarrow M \underset{R}{\otimes} S,$$

which represents multiplication by s.

Define a map

$$M \times S \longrightarrow M \underset{R}{\otimes} S,$$

by sending (m, t) to $m \otimes (st)$. It is not hard to see that this map is bilinear; by the universal property of the tensor product it descends to an R-linear map

$$M \underset{R}{\otimes} S \longrightarrow M \underset{R}{\otimes} S.$$

It is easy to check that under this rule for scalar multiplication $M \underset{R}{\otimes} S$ becomes an S-module. $\square$

Note that we can write down the rule for scalar multiplication on generators of $M \underset{R}{\otimes} S$,

$$(s, m \otimes s') \longrightarrow m \otimes (ss').$$

It is again interesting and informative to figure out what happens for vector spaces. For example, suppose that V is a real vector space. Then by extension of scalars, $W = V \underset{\mathbb{R}}{\otimes} \mathbb{C}$ is a complex vector space. Note that if $e_1, e_2, \dots, e_n$ is a basis of V, then $e_i \otimes 1$ is a basis of W, so that V and W have the same dimension.