HOMEWORK 9, DUE THURSDAY MARCH 14TH

1. Let R be an integral domain and let M be an R-module. We say that $m \in M$ is **torsion** if there is a non-zero element $r \in R$ such that $r \cdot m = 0$.
(i) Show that the subset T of all elements of M which are torsion is a submodule of M.
(ii) What are the torsion elements in
(a) $\mathbb{Q}/\mathbb{Z}$?
(b) $\mathbb{R}/\mathbb{Z}$?
(c) $\mathbb{R}/\mathbb{Q}$?
(iii) Is the $\mathbb{Z}$-module $\mathbb{Q}$
(a) torsion-free?
(b) free?
(c) finitely generated?
2. Let R be a PID and let M be a finitely generated module over R.
(i) Show that there is a free module F which is a quotient of M and which is maximal with respect to this property.
(ii) Show that there is an injective R-linear map $F \longrightarrow M$.
(iii) Show that the image of F is not always unique.
3. Let

$$A = \begin{pmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{pmatrix} \in M_{3,3}(\mathbb{Z}).$$

(i) Put A into Smith normal form D using elementary operations.
(ii) Check your answer using minors.
(iii) Explain how to find invertible matrices P and Q such that $D = QAP$.
4. Find the Smith normal form of

$$\begin{pmatrix} 2x-1 & x & x-1 & 1 \\ x & 0 & 1 & 0 \\ 0 & 1 & x & x \\ 1 & x^2 & 0 & 2x-2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x^2+2x & 0 & 0 & 0 \\ 0 & x^2+3x+2 & 0 & 0 \\ 0 & 0 & x^3+2x^2 & 0 \\ 0 & 0 & 0 & x^4+x^3 \end{pmatrix}$$

over the ring $\mathbb{R}[x]$.
5. Let G be the abelian group with presentation given by generators a, b and c, and relations $6a + 10b = 0$, $6a + 15c = 0$ and $10b + 15c = 0$. Determine the structure of G as a product of cyclic groups.

6. Let A be a complex square matrix with characteristic polynomial $(x+1)^6(x-2)^3$ and minimal polynomial $(x+1)^3(x-2)^2$. What are all of the possible Jordan normal forms for A?
7. Describe all conjugacy classes of the following finite groups. For each conjugacy class give the order and the minimal polynomial of an element.
(i)

$$\mathrm{GL}_2(\mathbb{F}_2)$$

(ii)

$$\mathrm{GL}_3(\mathbb{F}_2)$$

Challenge Problems: (Just for fun)

(iii)

$$\mathrm{SL}_2(\mathbb{F}_4)$$

the subgroup of $\mathrm{GL}_2(\mathbb{F}_4)$ of matrices with determinant one and

$$\mathbb{F}_4 = \frac{\mathbb{F}_2[x]}{\langle x^2+x+1\rangle} = \{\, 0, 1, \omega, \omega+1 \,\}$$

is the field with four elements.
8. Let R be a PID, let $F = R^n$ be a finitely generated free module over R of rank n and let $M \subset F$ be a free module. We are going to show that M is a free module over R of rank $m \leq n$.
Let $f\colon R^n \longrightarrow R$ be the projection onto the last factor and let G be the kernel. Let $N = M \cap G$.
(i) Show that G is a finitely generated module of rank $n-1$.
(ii) Show that N is a free module of rank l at most $n-1$.
(iii) Let $Q = f(M)$ be the image of M. Show that we may find $e \in M$ such that $f(e)$ generates Q.
(iv) Show that if $f_1, f_2, \dots, f_l$ are free generators of N then $f_1, f_2, \dots, f_l, e$ are free generators of M.
(v) Conclude that M is a free module of rank m at most n.
9. If A is a real $n \times n$ square matrix such that $A^2 + I_n = 0$ then show that $n = 2m$ is even and A is similar to the matrix in block form

$$\begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}.$$

10. Let R be the ring of all infinitely differentiable functions from $[-1, 1]$ to the real numbers $\mathbb{R}$. Show that R is not Noetherian.
11. Is there a 9×9 square matrix A such that A^2 has a Jordan form with blocks of size
(a) 4, 3 and 2?

(b) 4, 4 and 1?
(*Hint: If J is a Jordan block then what is the Jordan canonical form of J^2?*).