HOMEWORK 7, DUE THURSDAY FEBRUARY 29TH

1. Are the following true or false? If true prove them; if false give counterexamples.
(i) Every prime ideal is maximal.
(ii) Every ring is a PID.
(iii) Every UFD is a PID.
2. If

$$N_1 \subset N_2 \subset N_3 \subset \dots$$

is an ascending chain of submodules of an R-module M then show that the union N is a submodule.

Challenge Problems: (Just for fun)

3. Let

$$0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$$

be a short exact sequence of R-modules.
(a) Suppose that M is finitely generated. Show that N is finitely generated if and only if P is finitely generated (this gives another way to show that N is Noetherian if and only if M and P are Noetherian).
(b) Show that if P is free then

$$N \simeq M \oplus P.$$

In this case we say that the short exact sequence **splits**.
(c) Show that if P is not free then the short exact sequence does not necessarily split.
4. (a) Show the **five lemma** which states that if we have a commutative diagram

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

of R-modules with exact rows such that m and p are isomorphisms, l is surjective and q is injective then n is an isomorphism.
(*Hint: we can break this result into two four lemmas. One to show injectivity of* n *and the other to show surjectivity of* n).
(b) Use this to give another proof that in the proof of (11.6) that N_i is determined by M_i and P_i.