HOMEWORK 6, DUE THURSDAY FEBRUARY 22ND

1. Let M be an R-module and let $r \in R$. Show that the map

$$\phi \colon M \longrightarrow M \qquad \text{given by} \qquad m \longrightarrow rm$$

is R-linear.
2. Prove that a subset N of an R-module is a submodule if and only if it is non-empty and closed under addition and scalar multiplication.
3. Let $\phi \colon M \longrightarrow N$ be an R-linear map between two R-modules. Prove that the kernel of ϕ is a submodule of M.
4. Let M be an R-module. Prove that the intersection of any set of submodules is a submodule.
5. Let M be an R-module and let X be any subset of M. Prove the existence of the submodule generated by X.
6. Let M be an R-module and let X be any set. Show how the set of all maps from X to M becomes an R-module.
7. Let M and N be any two R-modules. Denote by $\mathrm{Hom}_R(M, N)$ the set of all R-linear maps from M to N. Show that this set is naturally an R-module.
8. Let M be an R-module and let X be a subset of M. The annihilator I of X, is the subset of all elements r of R, such that $rm = 0$, for all elements m of X. Show that I is an ideal of R. Prove also that the annihilator of X is equal to the annihilator of the submodule generated by X.

The next few results refer to the power series ring which is defined as follows. Let R be a commutative ring and let x be an indeterminate. The power series ring in R, denoted $R[\![x]\!]$, consists of all (possibly infinite) formal sums,

$$\sum_{n \geq 0} a_n x^n,$$

where $a_n \in R$. Thus if $R = \mathbb{Q}$, then both

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots,$$

and

$$1 + 2!x + 3!x^2 + 4!x^3 + \dots,$$

are elements of $\mathbb{Q}[\![x]\!]$, even though the second, considered as a power series in the sense of analysis, does not converge for any $x \neq 0$. Addition and multiplication of elements of $R[\![x]\!]$ are defined as for polynomials.

The degree of a power series is equal to the **smallest** n, so that the coefficient of a_n is non-zero. Even for a polynomial, in what follows the degree always refers to the degree as a power series.

9. (i) Show that $R[\![x]\!]$ is a ring.

(ii) Show that $f(x) \in R[\![x]\!]$ is invertible if and only if the degree of $f(x)$ is zero and the constant term is invertible. What is the inverse of $1 - x$?

(iii) Show that if R is an integral domain then the degree of a product is the sum of the degrees.

(iv) Show that if R is an integral domain then so is $R[\![x]\!]$.

(v) If F is a field then prove that $F[\![x]\!]$ is a Euclidean domain.

(vi) Show that if F is a field then $F[\![x]\!]$ is a UFD.

10. (i) See bonus problems.

(ii) Prove that if R is Noetherian then so is $R[\![x_1, x_2, \dots, x_n]\!]$, where the last term is defined appropriately.

Challenge Problems: (Just for fun)

10 (i). Show that if R is Noetherian then so is $R[\![x]\!]$.

11. Let M be a Noetherian R-module. If $\phi\colon M \longrightarrow M$ is a surjective R-linear map, prove that ϕ is an automorphism. (*Hint, consider the submodules,* $\mathrm{Ker}(\phi^n)$).