HOMEWORK 4, DUE THURSDAY FEBRUARY 8TH

1. Let R be an integral domain. Let a and b be two elements of R. Show that if d and d' are both a gcd for the pair a and b, then d and d' are associates.
2. Let R be a UFD.
(a) Prove that for every pair of elements a and b of R, we may find an element $m = [a, b]$ that is a **least common multiple**, that is

    (1) $a|m$ and $b|m$,
    (2) and if $a|m'$ and $b|m'$ then $m|m'$. w

Show that any two lcm's are associates.
(b) Show that if (a, b) denotes the gcd then $(a, b)[a, b]$ is an associate of ab.
3. Chapter 4, §5: 3(a), (d).
4. Find the greatest common divisor of $135 - 14i$ and $155 + 34i$ in the ring of Gaussian integers $\mathbb{Z}[i]$.
5. (a) Show that the elements 2, 3 and $1 \pm \sqrt{-5}$ are irreducible elements of $\mathbb{Z}[\sqrt{-5}]$.
(b) Show that every element of R can be factored into irreducibles.
(c) Show that R is not a UFD.

Challenge Problems: (Just for fun)

6. Let S be a commutative monoid, that is, a set together with a binary operation that is associative, commutative, and for which there is an identity, but not necessarily inverses. Treating this operation like multiplication in a ring, define what it means for S to have unique factorisation.
7. Let $v_1, v_2, \dots, v_n$ be a sequence of elements of $\mathbb{Z}^2$. Let S be the semigroup that consists of all linear combinations of $v_1, v_2, \dots, v_n$, with positive integral coefficients. Let the binary rule be ordinary addition. Determine which monoids have unique factorisation.
8. Show that there is a ring R, such that every element of the ring is a product of irreducibles, whilst at the same time the factorisation algorithm can fail.