MODEL ANSWERS TO THE NINTH HOMEWORK

1. For Chapter 2, Section 11: 1. Conjugacy in S_n is determined by cycle type. There are three conjugacy classes,

$$C_1 = \{e\} \qquad C_2 = \{(2,3),(1,3),(1,2)\} \qquad \text{and} \qquad C_3 = \{(1,2,3),(1,3,2)\}.$$

Everything commutes with the identity

$$C_e = S_3,$$

and the index of S_3 is one, the cardinality of C_1.
Consider $(1,2) \in C_2$. This only commutes with the identity and itself, so that

$$C_{(1,2)} = \{e,(1,2)\}.$$

The index is 3 which is the cardinality of C_2.
Consider $(1,2,3) \in C_2$. This only commutes with all of its powers

$$C_{(1,2,3)} = \{e,(1,2,3),(1,3,2)\}.$$

The index is 2 which is the cardinality of C_3.
There is only conjugacy class with one element. It follows that the centre is trivial and so the class equation reads

$$6 = |S_3| = |Z| + |C_2| + |C_3| = 1 + 3 + 2.$$

2. There are five conjugacy classes,

$$C_1 = \{I\}, \quad C_2 = \{R, R^3\}, \quad C_3 = \{R^2\}, \quad C_4 = \{F_1, F_2\} \quad \text{and } C_5 = \{D_1, D_2\}.$$

Everything commutes with the identity

$$C_I = D_4,$$

and the index of D_4 is one, the cardinality of C_1. Everything also commutes with R^2,

$$C_{R^2} = D_4,$$

and the index of D_4 is one, the cardinality of C_3.
Consider $R \in C_2$. This only commutes with all of its powers

$$C_R = \{I, R, R^2, R^3\}.$$

The index is 2 which is the cardinality of C_2.
Now consider $F_1 \in C_4$. This commutes with

$$C_{F_1} = \{I, R^2, F_1, F_2\}.$$

The index is 2 which is the cardinality of C_4.

Finally consider $D_1 \in C_5$. This commutes with

$$C_{D_1} = \{I, R^2, D_1, D_2\}.$$

The index is 2 which is the cardinality of C_5.
There are two conjugacy classes with one element. It follows that the centre

$$Z = \{I, R^2\}$$

has two elements. In this case the class equation reads

$$8 = |D_4| = |Z| + |C_2| + |C_4| + C_5| = 2 + 2 + 2 + 2.$$

3. Follows from (4).
4. Done in class.
5. There are two ways to prove this.
Recall that Z is normal in G. As the quotient G/Z has order 1 or p, the quotient is cyclic. But then G is abelian.
Aliter: Note that if $c \in G$ then $Z \subset C(c)$, the centraliser.
As Z has index at most p there are two cases, $Z = C(c)$ or $C(c) = G$.
Suppose that $Z = C(c)$. In this case $c \in Z$ so that c commutes with everything.
Suppose that $C(c) = G$. In this case c commutes with everything.
Either way, an arbitrary element c of G commutes with everything and G is abelian.
10. Conjugation by x defines an inner automorphism of G,

$$\phi\colon G \longrightarrow G \qquad \text{given by} \qquad \phi(g) = xgx^{-1}.$$

We want to show

$$N(\phi(H)) = \phi(N(H))).$$

Suppose that y belongs to the RHS. We show that it belongs to the LHS.
By assumption $y = \phi(n)$, where $n \in N(H)$. Suppose that $k \in \phi(H)$. Then we may find $h \in H$ such that $k = \phi(h)$. We have

$$\begin{aligned} yky^{-1} &= \phi(n)\phi(h)\phi(n^{-1}) \\ &= \phi(nhn^{-1}). \end{aligned}$$

Now $l = nhn^{-1} \in H$ as $h \in H$ and $n \in N(H)$. Thus $\phi(l) \in \phi(H)$ and so $y \in N(\phi(H))$ as k is arbitrary.
It follows that

$$N(\phi(H)) \supset \phi(N(H))).$$

We have shown that

$$N(\psi(K)) \supset \psi(N(K)))$$

for any subgroup K and any inner automorphism ψ. Let ψ be the inverse of ϕ and let $K = \phi(H)$. Then

$$N(H) \supset \psi(N(\phi(H))) \qquad \text{as} \qquad H = \psi(K).$$

Applying ϕ to both sides gives

$$N(\phi(H)) \subset \phi(N(H)).$$

Thus the LHS is contained in the RHS.
16. $36 = 2^2 \cdot 3^2$. Thus there is a Sylow 3-subgroup H of order 9. Let S be the set of left cosests. Define an action of G on S as follows

$$G \times S \longrightarrow S \qquad \text{given by} \qquad g \cdot (aH) = (ga)H.$$

This gives rise to a homomorphism

$$\phi\colon G \longrightarrow A(S).$$

What can we say about the kernel N of ϕ? If g is in the kernel then $\phi(g)$ fixes H, that is, it stabilises H. Thus $gH = g \cdot H = H$ and so $g \in H$. Thus

$$\operatorname{Ker} \phi = N \subset H.$$

H has index 4 and so S has four elements. Thus $A(S) \simeq S_4$ has 24 elements. The image G' of G is a subgroup of $A(S)$. 9 does not divide 24 and so 9 does not divide G' by Lagrange. Thus the kernel must be non-trivial.
The kernel is a subgroup of H and so N is a normal subgroup of order 3 or 9.
17. $108 = 2^2 \cdot 3^3$. Thus there is a Sylow 3-subgroup H of order 27. As before this gives us a group homomorphism

$$\phi\colon G \longrightarrow S_4.$$

Let N be the kernel. Then N is a normal subgroup of G and $\phi(G)$ is isomorphic to G/N. As 27 divides G but 9 does not divide $24 = 4!$ it follows that the order of N is divisible by 9. Thus N is a normal subgroup of order 9 or 27.
2. (i) True.
Let $\phi\colon G \longrightarrow G$ be an automorphism of G. We have to show that $\phi(K) \subset K$.
As H is characteristically normal in G it follows that $\phi(H) \subset H$. Define a function

$$\psi\colon H \longrightarrow H \qquad \text{by the rule} \qquad \psi(h) = \phi(h) \in H.$$

It is easy to see that ψ is a group homomorphism, as ϕ is a group homomorphism.

By assumption ϕ has an inverse ρ. $\rho(H) \subset H$ and so we may define a function

$$\sigma\colon H \longrightarrow H \qquad \text{by the rule} \qquad \sigma(h) = \rho(h) \in H.$$

By what we already observed, σ is a group homomorphism and σ is clearly the inverse of ψ.
Thus ψ is an automorphism of H. As K is characteristically normal in H it follows that

$$\begin{aligned}\phi(K) &= \psi(K)\\ &\subset K.\end{aligned}$$

Thus K is characteristically normal in G.
(ii) False. Let

$$G = A_4, \quad H = \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\} \quad \text{and} \quad K = \{e, (1,2)(3,4)\}.$$

Then the elements of H consist of the identity and all of the elements of order 2. It follows that H is characteristically normal in A_4. K has index two in H and so K is normal in H.
However K is not normal in G. Let $\sigma = (1,2)(3,4) \in K$ and let $\tau = (1,2,3)$. Then

$$\tau\sigma\tau^{-1} = (2,3)(1,4) \notin K.$$

(iii) True. Let $g \in G$. As observed in (i) we get an automorphism of H by the rule

$$\phi\colon H \longrightarrow H \qquad \text{given by} \qquad h \longrightarrow ghg^{-1}$$

As K is characteristically normal in H we have

$$\begin{aligned}gKg^{-1} &= \phi(K)\\ &\subset K.\end{aligned}$$

Therefore K is normal in G.
(iv) False. If H is characteristically normal in G then it is certainly normal. Thus (ii) provides a counterexample.
3. Suppose that $G = \langle a \rangle$ is cyclic and let ϕ be an automorphism of G. Then ϕ is determined by what it does to a and it must send a to another generator of G. We have already seen that there is a unique isomorphism sending the generator of a cyclic generator to a generator. If G is infinite then the only generators of G are a and a^{-1}. Thus the automorphism group of G has two elements and $\text{Aut}(G) \simeq \mathbb{Z}_2$.
Now suppose that a has order n. a^i is a generator of G if and only if i is coprime to n. This defines a function

$$f\colon \text{Aut}(G) \longrightarrow U_n \qquad \text{given by} \qquad f(\phi) = i,$$

where $\phi(a) = a^i$. We check that f is a group homomorphism. Suppose that α and β are two automorphisms of G and let $\gamma = \alpha\beta$. Suppose that $\beta(a) = a^i$ and $\alpha(a) = a^j$. We have

$$\begin{aligned}
\gamma(a) &= (\alpha\beta)(a) \\
&= (\alpha \circ \beta)(a) \\
&= \alpha(\beta(a)) \\
&= \alpha(a^j) \\
&= \alpha(a)^j \\
&= (a^i)^j \\
&= a^{ij}.
\end{aligned}$$

Thus

$$\begin{aligned}
f(\gamma) &= ij \\
&= f(\alpha)f(\beta),
\end{aligned}$$

so that f is an isomorphism.

4. Let G be a group of order n. If n is prime then G is cyclic.

So we may assume that $n = 14$. If G is abelian then G is isomorphic to

$$G \simeq \mathbb{Z}_2 \times \mathbb{Z}_7 \simeq \mathbb{Z}_{14}.$$

Suppose that G is not abelian. The order of an element must be a divisor of 14, so that the possible orders are 1, 2, 7 or 14. G is not cyclic, as it is not abelian and so there are no elements of order 14. There is only one element of order 1, the identity. As G is not abelian, not every element has order 2.

It follows that there is an element a of order 7. Let H be the group it generates. Then H has index two in G and so H is normal in G. G/H is a group of order two. Pick $b \in G \setminus H$. The image of b in G/H has order 2. Thus the order of b is divisible by 2. Thus the order of b is two, as it is not 14.

Note that $G = \langle a, b \rangle$, as the order of the group generated by a and b is divisible by both 2 and 7.

At this point we can proceed as in the lecture notes. Here is an alternative and more robust way to proceed.

Conjugation by b induces an automorphism of G. It restricts to an automorphism ϕ of H, as H is normal

$$\phi \colon H \longrightarrow H \qquad \text{given by} \qquad \phi(h) = bhb^{-1} \in H.$$

As G is not abelian, $bab^{-1} \neq a$ and so ϕ is non-trivial. Thus ϕ has order 2 as b has order 2.

By question (3),

$$\mathrm{Aut}(H) = U_7.$$

The elements of U_7 are the integers from 1 to 6. $2^2 = 4$ and $2^3 = 8 = 1 \mod 7$. Thus 2 is an element of order 3. $3^2 = 9 = 2 \mod 7$ so that 3 is an element of order 6. It follows that $U_7 = \langle 3 \rangle$ is cyclic and there is unique element of order 2, $3^3 = 27 = 6 \mod 7$.
In this case

$$\phi(a) = a^6 = a^{-1} \qquad \text{so that} \qquad bab^{-1} = a^{-1}.$$

Thus $G \simeq D_7$, the Dihedral group of order 14.
Thus there are two groups of order 14, the cyclic group and the Dihedral group.
Challenge Problems (Just for fun)
5. $n = 12 = 2^2 \cdot 3$. If G is abelian then G is isomorphic to either

$$\mathbb{Z}_4 \times \mathbb{Z}_3 \simeq \mathbb{Z}_{12} \qquad \text{or} \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \simeq \mathbb{Z}_2 \times \mathbb{Z}_6.$$

Suppose that G is not abelian.
TBC ....
6. There are many ways to proceed; the main point is to try to be reasonably efficient. Suppose that $61 \leq n \leq 167$. By Sylow's theorem, we may assume that n is not of the form mp^k, where $m < p$ and p is prime (note that this includes the case when n is either a power of a prime or the product of two primes).
This leaves

$$63, 70, 72, 80, 84, 90, 96, 105, 108, 112, 120, 126, 132, 135, 140, 144, 150, 154, 160, 165.$$

Suppose that n has the form $p^\alpha q^\beta$, where $p < q$. Then the number of Sylow q-subgroups is equal to

$$1, p, p^2, p^3, \dots.$$

If this is not equal to 1, it must be greater than $q > p$, so that there must be at least p^2 such subgroups. Suppose that there are p^2. Then q divides $p^2 - 1 = (p-1)(p+1)$. It follows that $q = p + 1$, so that $p = 2$ and $q = 3$. Otherwise, the number of such subgroups is at least p^3. If $\beta = 1$, this gives at least $p^3(q-1)$ elements of order q. If $\alpha = 3$, then there are only p^3 elements left and there is then a unique Sylow p-subgroup.
It follows that if n is not of the form $2^\alpha 3^\beta$, and either $\alpha \leq 2$ or $\beta = 1$ and $\alpha = 3$, then G is not simple. Thus we can eliminate all such numbers. This leaves:

$$70, 72, 84, 90, 96, 105, 108, 120, 126, 132, 140, 144, 150, 154, 165.$$

Now suppose that the largest prime dividing n is eleven. Let x be the number of Sylow 11-subgroups. Then there are at least $10x$ elements of G whose order is a power of 11. Since $10x < 168$, we see that $x \leq 16$. But then

$$x = 1, 12.$$

Thus assuming that $x \neq 1$, n is divisible by 12.
Now suppose that the largest prime dividing n is seven. Let x be the number of Sylow 7-subgroups. Since $6x < 168$, we see that $x < 28$. But then

$$x = 1, 8, 15, 22.$$

Thus assuming that $x \neq 1$, then either n is divisible by 2^3 or $3 \cdot 5$.
This leaves

$$72, 80, 90, 96, 105, 108, 112, 120, 132, 144, 150, 160.$$

We handle the rest using a case by case analysis. Note first that if we can find a nontrivial group homomorphism

$$\rho \colon G \longrightarrow S_k,$$

where either n does not divide $k!$, or $n = k!$, then we are done. Indeed we may assume that the kernel is trivial, in which case ρ is injective, so that by Lagrange $G \simeq S_k$. But then A_k is a proper normal subgroup.
Note that G acts on the Sylow p-subgroups and that G acts on the left cosets of any subgroup H. So the number of Sylow p-subgroups and the index of any subgroup is at most 4, at most 5 if n does not divide 120, or $n = 120$, and at most 6, if n does not divide 720.
Suppose that $n = 72 = 2^3 \cdot 3^2$. Let x be the number of Sylow 3-subgroups. Then

$$x = 1, 4, 7, \dots$$

and x divides 8. But then $x \leq 4$ and we are done.
If $n = 80 = 2^4 \cdot 5$, then a Sylow 2-subgroup has index 5 and we are done.
If $n = 90 = 2 \cdot 3^2 \cdot 5$, then let x be the number of Sylow 5-subgroups. Then

$$x = 1, 6, 11, 16,$$

and divides 18. Thus we may assume that $x = 6$, and there are 30 elements of order 5. Now let y be the number of Sylow 3-subgroups. Then

$$y = 1, 4, 7, 10,$$

and divides 10. Thus we may assume that $y = 10$. Suppose that every two Sylow 3-subgroups intersect only in the trivial group. Then there are $10 \cdot 8 = 80$ elements which belong to these groups, which is far too many. The only possibility is that there are two Sylow 3-subgroups P and Q such that $|P \cap Q| = 3$. Let N be the normaliser of $H = P \cap Q$. As H is normal in P and Q,

$$N \supset \langle P, Q \rangle \supset PQ.$$

Thus N contains at least

$$\frac{|P||Q|}{|H|} = 27,$$

elements. Thus the index of N is at most three, and we are done.
If $n = 96 = 2^5 \cdot 3$, then the number of Sylow 2-subgroups divides 3, and we are done.
If $n = 105 = 3 \cdot 5 \cdot 7$, let x be the number of Sylow 7-subgroups. Then

$$x = 1, 8, 15,$$

and divides 15. Thus we may assume that $x = 15$. But then there are $15 \cdot 6 = 90$ elements of order 5. Let y be the number of Sylow 5-subgroups. Then

$$y = 1, 6, 11, 16, 21,$$

and divides 21. Thus we may assume that $y = 21$, so that there are $21 \cdot 4$ elements of order five, impossible.
If $n = 108 = 2^2 \cdot 3^3$, then the index of a Sylow 3-subgroup Q is 4 and we are done.
If $n = 120 = 2^3 \cdot 3 \cdot 5$, then let x be the number of Sylow 5-subgroups. Then

$$x = 1, 6, 11, 16, 21, \dots,$$

and x divides 24. But then $x = 6$, and there are 24 elements of order 5. Let y be the number of Sylow 3-subgroups. Then

$$y = 1, 4, 7, 10, \dots,$$

and y divides 40. Thus $y = 10$, and there are 20 elements of order 3. Let z be the number of Sylow 2-subgroups. Then

$$z = 1, 3, 5, 7, \dots,$$

and z divides 15. Thus we may assume that $z = 15$. If every pair of Sylow 2-subgroups have trivial intersection, then there are $15 \cdot 7 = 105$ elements belonging to these subsets, which is impossible.
Otherwise there is a pair P and Q of Sylow 2-subgroups, which intersect $H = P \cap Q$ non-trivially. Let N be the normaliser of H. If $|N| > 8$, then the index of N is at most five, and we are done.

If $n = 122 = 2^4 \cdot 7$, then we may assume that there are at least 2 Sylow 2-subgroups P and Q. Let $H = P \cap Q$. If $|H| \leq 2$, then

$$|G| \geq |PQ| = \frac{|P||Q|}{|H|} \geq 128,$$

a contradiction. Thus $|H| \geq 4$. Let $P' \subset P$ and $Q' \subset Q$ be two subgroups, in which H has index two. Then H is normal in both P' and Q' so that the normaliser N of H contains both P' and Q'. But then

$$|N| \geq 16.$$

If $|N| = 16$, then H is normal in a Sylow 2-subgroup, so that N contains both P and Q, a contradiction. But then $|H| > 16$, and its index is at most 4, and we are done.
If $n = 132 = 2^2 \cdot 3 \cdot 11$, then let x be the number of Sylow 11-subgroups. Then

$$x = 1, 12, \dots,$$

and divides 12. Thus we may assume that $x = 12$. In this case there are $12 \cdot 10 = 120$ elements of order 11. Let y be the number of Sylow 3-subgroups. Then

$$y = 1, 4, 7, \dots,$$

and y divides 44. But then we may assume that $y \geq 22$ and so there would be at least $22 \cdot 2 = 44$ elements of order 3, impossible.
If $n = 144 = 2^4 \cdot 3^2$, then let x be the number of Sylow 3-subgroups. Then

$$x = 1, 4, 7, \dots,$$

and x divides 16. But then we may assume that $x = 16$. If every pair of Sylow 3-subgroups have trivial intersection, then there are $16 \cdot 8 = 128$ elements belonging to these subgroups. The remaining 16 elements must form the unique Sylow 2-subgroup. Otherwise the intersection H of two Sylow 3-subgroups P and Q has cardinality 3. In this case the normaliser N of H has order at least 27, so that its index is at most four and we are done.
If $n = 150 = 2 \cdot 3 \cdot 5^2$, then a Sylow 5-subgroup has index six, and we are done.
If $n = 160 = 2^5 \cdot 5$. Then a Sylow 2-subgroup has index 5 and we are done.