MODEL ANSWERS TO THE EIGHTH HOMEWORK

1. For Chapter 3, Section 3: 1. (a)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 5 & 1 & 3 & 7 & 8 & 9 & 6 \end{pmatrix} = (1,2,4)(3,5)(6,7,8,9),$$

a product of $2+1+3=6$ transpositions. Even.
(b) $5+2=7$, odd.
(c) $5+5=10$, even.
(d) $1+2+1+2+2=8$, even.
1. For Chapter 3, Section 3: 2. We have already seen that a k-cycle is a product of $k-1$ transpositions. Thus the result is clear.
1. For Chapter 3, Section 3: 3. Clear, since they have the same cycle type.
1. For Chapter 3, Section 3: 5. First the bottom row is missing 4 and 5. Let us try

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 2 & 4 & 5 & 7 & 8 & 9 & 6 \end{pmatrix} = (1,3,2)(6,7,8,9).$$

This is a product of $5=2+3$ tranpositions. Thus this permutation is odd. Now consider

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 2 & 5 & 4 & 7 & 8 & 9 & 6 \end{pmatrix}.$$

This differs from the first permutation by a transposition. Hence it is even, as the other is odd.
1. For Chapter 3, Section 3: 6. Since every element of A_n is a product of an even number of transpositions, pairing off the transpositions arbitrarily, it suffices to prove that the product of two transpositions is a product of 3-cycles.
Suppose that (a,b) and (c,d) are two transpositions. There are three cases. If $\{a,b\}=\{c,d\}$, then their product is the identity, which is the product of any 3-cycle with its inverse.
Now suppose that $\{a,b\}\cap\{c,d\}$ contains one element, say a. Then

$$(a,b)(a,c)=(a,c,b)$$

a 3-cycle.
Finally suppose that $\{a,b\}\cap\{c,d\}$ is empty. Then

$$(a,b)(c,d)=(a,b)(a,c)(a,c)(c,d)=(a,c,b)(c,d,a),$$

a product of two 3-cycles.
1. For Chapter 3, Section 3: 7. As every element of A_n is a product of 3-cycles, it suffices to prove that every 3-cycle is a product of n-cycles. By symmetry (in fact conjugation by elements of S_n) it suffices to prove that there is a single 3-cycle, which is a product of n-cycles. Consider the elements $\sigma = (1, 2, 3, 4, \dots, n)$ and $\tau = (1, 3, 2, n, n-1, \dots, 4)$. In this case

$$\sigma\tau = (1, 4, 2),$$

as is easily seen by direct computation.
1. For Chapter 3, Section 3: 8. Done in class. Let

$$V = \{e, (1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}.$$

Then V is easily seen to be closed under products and inverses. As V is a union of conjugacy classes, it follows that V is normal (not only in A_4, but even in S_4 in fact).
Challenge Problems (Just for fun)
2. For Chapter 3, Section 3: 9. This is a little tricky and there are many different ways to prove this result. Let N be a normal subgroup of A_n and suppose that $N \neq \{e\}$.
Suppose that $\alpha \neq e$ is an element of N, that moves the fewest number of elements. We try to prove that α is in fact a 3-cycle. Suppose not; we will derive a contradiction.
First suppose that in the cycle decomposition of α, we have a cycle of length greater than two. Putting this first and relabelling, we have

$$\alpha = (1, 2, 3, \dots) \dots$$

We may also assume that α does not fix 4. Consider conjugating α by $\beta = (1, 2)(3, 4)$ to get α_1. Then

$$\alpha_1 = (2, 1, 4, \dots).$$

Note that

$$\gamma = \alpha_1\alpha = \beta\alpha\beta^{-1}\alpha \in N,$$

as N is normal. γ fixes everything that α fixes and also fixes 1, a contradiction, as α is supposed to fix the most elements.
Therefore α is a product of disjoint transpositions:

$$\alpha = (1, 2)(3, 4) \dots.$$

Consider conjugating α by $\beta = (3, 4, 5)$ to get α_1. Then

$$\alpha_1 = (1, 2)(4, 5) \dots,$$

Then

$$\gamma = \alpha_1\alpha = (3, 4)(4, 5) \dots \in N,$$

which fixes 1 and 2 and fixes everything α fixes except possibly 5, a contradiction.
Thus α is a 3-cycle.
For Chapter 3, Section 3: 10. This follows *almost* immediately from 9 and 6. In S_n, the conjugacy classes are determined by cycle type. In particular any two 3-cycles are conjugate in S_n. We want to show that they are conjugate in A_n. Now suppose that σ and σ' are two 3-cycles and that $\tau \in S_n$ conjugates σ to σ'. If τ is even there is nothing to prove. Otherwise pick a transposition τ_1 that is disjoint from the numbers that σ permutes (possible as $n \geq 5$). Then $\tau\tau_1$ is even and is easily seen to still conjugate σ to σ'.
Let N be a normal subgroup of A_n. Suppose that $N \neq \{e\}$. Then by the previous question N must contain a 3-cycle. As N is normal, it must contain every 3-cycle, as any two 3-cycles are conjugate. But then by 6, $N = A_n$.