MODEL ANSWERS TO THE SEVENTH HOMEWORK

1. For Chapter 2, Section 9: 1. Let $\phi\colon G_1 \times G_2 \longrightarrow G_2 \times G_1$ be the homomorphism that sends (g_1, g_2) to (g_2, g_1). This is clearly a bijection. We check that it is a homomorphism. Suppose that (g_1, g_2) and $(h_1, h_2) \in G_1 \times G_2$. Then

$$\begin{aligned}\phi((g_1, g_2)(h_1, h_2)) &= \phi(g_1h_1, g_2h_2)\\ &= (g_2h_2, g_1h_1)\\ &= (g_2, g_1)(h_2, h_1)\\ &= \phi(g_1, g_2)\phi(h_1, h_2).\end{aligned}$$

Thus ϕ is a homomorphism.
Alternatively, we could use the universal property of the product. Both $G_1 \times G_2$ and $G_2 \times G_1$ satisfy the universal properties of a product and so they must be isomorphic, by uniqueness.
1. For Chapter 2, Section 9: 2. These properties are clearly preserved by isomorphism, so we may as well assume that $G_1 = \mathbb{Z}_m$ and $G_2 \simeq \mathbb{Z}_n$. Consider $(1, 1) \in G_1 \times G_2$. Suppose that $k(1, 1) = (0, 0)$. Then $k = 0 \mod m$ and $k = 0 \mod n$. As m and n are coprime it follows that $k = 0 \mod mn$. But then the order of $(1, 1)$ is at least mn. As $G_1 \times G_2$ is a group of order mn, it follows that $G_1 \times G_2$ is cyclic, generated by $(1, 1)$.
Now suppose that m and n are not coprime. Suppose that $l = mn/d$, where d is a non-trivial divisor of both m and n (for example the gcd). Pick $(a, b) \in \mathbb{Z}_m \times \mathbb{Z}_n$. Then $l(a, b) = (la, lb)$. But la is divisible by m and so $la = 0 \mod m$ and lb is divisible by n so that $lb = 0 \mod n$. But then the order of (a, b) is at most l and $G_1 \times G_2$ is certainly not cyclic.
1. For Chapter 2, Section 9: 3. Define a homomorphism

$$\phi\colon G \longrightarrow T$$

by the rule $\phi(g) = (g, g)$. We check that this is a homomorphism. Suppose that g and $h \in G$. Then

$$\begin{aligned}\phi(gh) &= (gh, gh)\\ &= (g, g)(h, h)\\ &= \phi(g)\phi(h).\end{aligned}$$

Thus ϕ is a homomorphism. ϕ is clearly a bijection and so it is an isomorphism.
Suppose that T is normal. Pick a and b in G. Then $(a, a) \in T$ and the conjugate of this element by (b, e) is also in T. Thus

$$(b, e)(a, a)(b, e)^{-1} = (bab^{-1}, a) \in T$$

As this is an element of T, we have $bab^{-1} = a$ so that $ba = ab$. As a and b are arbitrary, G is abelian.
Now suppose that G is abelian. Pick $(g, g) \in T$ and $(a, b) \in G \times G$. Then

$$\begin{aligned} (a, b)(g, g)(a, b)^{-1} &= (aga^{-1}, bgb^{-1}) \\ &= (gaa^{-1}, gbb^{-1}) \\ &= (g, g). \end{aligned}$$

Thus T is normal.
2. Let $h \in H$ and $k \in K$ and let $a = hkh^{-1}k^{-1}$. As K is normal, $hkh^{-1} \in K$, so that $a = (hkh^{-1})k^{-1} \in K$. On the other hand, as H is normal $kh^{-1}k^{-1} \in H$ and so $a = h(kh^{-1}k^{-1}) \in H$. Thus $a \in H \cap K$ and so $a = e$. Thus $hk = kh$ and h and k commute.
3. Suppose that G is isomorphic to $G' \times H'$. Then we might as well assume that $G = H' \times K'$. In this case take $H = H' \times \{f\}$ and $K' = \{e\} \times K$, where e is the identity of H' and f is the identity of K'. Let p be the projection of G down to H'. Then p is a homomorphism, since this is part of the defining property of a categorical product. The kernel is K, so that K is normal in G. Similarly H is normal in G.
Define a homomorphism

$$\phi \colon H' \longrightarrow H$$

by sending h to (h, e). ϕ is clearly an isomorphism. Similarly K is isomorphic to K'. Hence the first property.
Suppose that $(a, b) \in H \cap K$. Then $a = e$ and $b = f$ so that $(a, b) = (e, f)$ is the identity of G. Hence the second property.
Suppose that $(h', k') \in G$, where $h' \in H'$ and $k' \in K'$. Then $(h', k') = (h', f)(e, k') = hk$ where $h = (h', f) \in H$ and $k = (e, k') \in K$. Thus $(h', k') \in H \vee K$ and $G = H \vee K$. Hence the third property.
Now suppose that (1)-(3) hold. Since H and K generate G, every element of G is a product of elements of H and K. As H and K are normal in G, the elements of H commute with the elements of K. Thus it is easy to prove that HK is closed under products and inverses and it follows that every element of G is of the form hk so that $G = HK$.

Define a homomorphism

$$\phi\colon G \longrightarrow H \times K,$$

by sending $g = hk$ to (h, k). Suppose that $h_1k_1 = h_2k_2$. Then $h_2^{-1}h_1 = k_2k_1^{-1} \in H \cap K$. Thus $h_2^{-1}h_1 = k_2k_1^{-1} = e$, the identity of G. Thus $h_1 = h_2$ and $k_1 = k_2$. Thus ϕ is well-defined.

The composition of ϕ with the two projection maps are the two identities, and these are both homomorphisms. By the universal property of a product, it follows that ϕ is a homomorphism.

ϕ is clearly surjective, and it is injective, as the kernel is clearly trivial. Thus ϕ is an isomorphism and G is isomorphic to $H \times K$. But $H \times K$ is clearly isomorphic to $H' \times K'$ and so we are done.

Challenge Problems (Just for fun)

4. (i) The direct sum is similar to the product, except that all the arrows go the other way.

The **direct sum** of two objects X and Y is an object Z together with two morphisms $i\colon X \longrightarrow Z$ and $j\colon Y \longrightarrow Z$ which are universal amongst all such morphisms:

Suppose that there are morphisms $f\colon X \longrightarrow W$ and $g\colon Y \longrightarrow W$. Then there is a unique morphism $Z \longrightarrow W$ which makes the following diagram commute,

$$\begin{tikzcd}
X \arrow[d, "i"'] \arrow[dr, "f"] & \\
Z \arrow[r, dashed] & W \\
Y \arrow[u, "j"] \arrow[ur, "g"'] &
\end{tikzcd}$$

(ii) The direct sum of two sets X and Y is the disjoint union $X \coprod Y$. The two functions i and j are the obvious inclusions.

(iii) If G and H are two abelian groups then the product $G \times H$ is also the direct sum. The two group homomorphisms i and j are

$$i\colon G \longrightarrow G \times H \qquad \text{given by} \qquad g \longrightarrow (g, f)$$
$$j\colon H \longrightarrow G \times H \qquad \text{given by} \qquad h \longrightarrow (e, h).$$