MODEL ANSWERS TO THE SIXTH HOMEWORK

1. Chapter 3, Section 6: 7. As G is cyclic, G is generated by a single element a. But then G/N is generated by $u(a) = aN$.
Chapter 3, Section 6: 8. Pick two elements of G/N. As G/N is the set of left cosets in G, these two elements have the form aN and bN. It follows that

$$\begin{aligned}(aN)(bN) &= abN \\ &= baN \\ &= (bN)(aN),\end{aligned}$$

where we use the fact that G is abelian to deduce $ab = ba$.
But then G/N is abelian.
Chapter 3, Section 6: 11. Suppose that G/Z is cyclic. Then there is an element a of G such that aZ generates G/Z, so that every left coset has the form a^iZ, for some i. Pick two elements x and y of G. Then $xZ = a^iZ$ and $yZ = a^jZ$, for some i and j, so that $x = a^iz_1$ and $y = a^jz_2$, for $z_i \in Z$, $i = 1, 2$.
Then

$$\begin{aligned}xy &= (a^iz_1)(a^jz_2) \\ &= a^ia^jz_1z_2 \\ &= a^{i+j}z_1z_2.\end{aligned}$$

Similarly $yx = a^{i+j}z_1z_2$. Thus $xy = yx$ and G is abelian.
Chapter 3, Section 6: 12. Suppose that G/N is abelian. Pick a and $b \in G$. Then

$$\begin{aligned}abN &= aNbN \\ &= bNaN \\ &= baN,\end{aligned}$$

so that $abN = baN$ and so $ab = ban$ for some $n \in N$. It follows that

$$\begin{aligned}n &= (ba)^{-1}ab \\ &= a^{-1}b^{-1}ab,\end{aligned}$$

so that $a^{-1}b^{-1}ab \in N$.

Chapter 3, Section 7: 2. We want to use the First Isomorphism Theorem. Define a function

$$\phi\colon G \longrightarrow \mathbb{R}$$

by sending f to $\phi(f) = f(1/4)$. Suppose that f and $g \in G$. Then

$$\begin{aligned}\phi(f+g) &= (f+g)(1/4)\\ &= f(1/4) + g(1/4)\\ &= \phi(f) + \phi(g).\end{aligned}$$

Thus ϕ is a homorphism. ϕ is clearly surjective. For example, given a real number a, let f be the constant function $f(x) = a$. Then $\phi(f) = f(1/4) = a$.
The kernel of ϕ consists of all functions that vanish at $1/4$, that is, N. Thus by the First Isomorphism Theorem, $G/N \simeq \mathbb{R}$.
Chapter 3, Section 7: 4. We first prove (a) and (c). Define a homomorphism

$$\phi\colon G \longrightarrow G_2,$$

by sending $g = (g_1, g_2)$ to g_2. Suppose that $g = (g_1, g_2)$ and $h = (h_1, h_2)$ are in G. Then

$$\begin{aligned}\phi(gh) &= \phi(g_1h_1, g_2h_2)\\ &= g_2h_2\\ &= \phi(g_1, g_2)\phi(h_1, h_2)\\ &= \phi(g)\phi(h).\end{aligned}$$

Thus ϕ is a homomorphism. ϕ is clearly surjective as given $g_2 \in G$, $\phi(e_1, g_2) = g_2$.
Suppose that $(g_1, g_2) \in \operatorname{Ker}\phi$. Then $g_2 = e_2$. Thus $N = \operatorname{Ker}\phi$. Hence (a). (c) follows from the First Isomorphism Theorem. To prove (b), define a homomorphism

$$f\colon N \longrightarrow G_1$$

by sending (g_1, e_2) to g_1. This is clearly an isomorphism.
Chapter 3, Section 7: 6. By definition the order of a is the order of the subgroup $H = \langle a\rangle$ and the order of aN is the order of the subgroup $H' = \langle aN\rangle$. Now it is clear that H' is the image of H under the canonical homomorphism

$$u\colon G \longrightarrow G/N.$$

So it suffices to prove that if we have a surjective homomorphism

$$\phi\colon H \longrightarrow H'$$

then the order of H' divides the order of H. But by the first isomorphism Theorem,

$$H' \simeq H/H'',$$

where H'' is the kernel of ϕ. Thus the order of H' is the index of H'' in H, the number of left cosets of H'' in H, which by Lagrange divides the order of H.