24. Finite matrix groups

The aim of this section is to classify finite subgroups H of $G = \mathrm{GL}(3, \mathbb{R})$, the group of 3×3 invertible matrices with real entries.

Note that G acts on $\mathbb{R}^3$ in the obvious way, via matrix multiplication

$$G \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \qquad \text{given by} \qquad g \cdot v = gv.$$

Suppose we are given a finite subgroup H of G. We will use the fact that

Lemma 24.1. *Every finite subgroup of* $\mathrm{GL}(n, \mathbb{R})$ *is conjugate to a subgroup of* $\mathrm{O}(n, \mathbb{R})$*, the group of all orthogonal matrices.*

We will prove this later. Note that the orthogonal group preserves distances between vectors. Thus we may assume that H preserves distances.

Note that there is a natural map

$$\phi \colon G \longrightarrow \mathbb{R}^* \qquad \text{given by} \qquad g \longrightarrow \det(g).$$

We have

$$\begin{aligned} \phi(gh) &= \det(gh) \\ &= \det(g)\det(h) \\ &= \phi(g)\phi(h), \end{aligned}$$

so that ϕ is a group homomorphism. Orthogonal matrices have determinant ± 1.

Let $v \in \mathbb{R}^3$ be a non-zero vector and consider the orbit of v. We pick v so that the linear span of the orbit has maximal dimension. There are three cases.

The orbit of v lies in a line. We might as well take this line to be the x-axis. If $v = (1, 0, 0)$ then $g \cdot v = \pm v$. Thus H is a group of order at most two. Thus H is either the trivial group or the group $\pm I_3$, isomorphic to $\mathbb{Z}_2$. If we consider only transformations of determinant one then we only get the identity.

Suppose that the orbit is planar, that is, the orbit lies in a plane. We might as well take this plane to be the plane $z = 0$. In this case the orbit of v lies on a circle of radius $|v| = r$. Thus we get a regular n-gon, for some $n \geq 3$.

If g fixes two non-collinear vectors in the plane $z = 0$ then $g = I_3$. Thus the action of H on the regular n-gon induces an embedding into the symmetries of a regular n-gon. Thus H is isomorphic to a subgroup of D_n. Rotations have determinant one and flips determinant -1.

By definition H acts transitively on the vertices of the n-gon. Thus the order of H is at least n. If the order is $2n$ then $H = D_n$. Suppose that the order is n. Pick a vertex. Then the stabiliser must be trivial, as the orbit of the vertex is equal to the order of H.

One obvious possiblity is that H consists of all rotations. Otherwise H must contain flips. If n is odd then any flip fixes a vertex. Thus $n = 2m \geq 4$ is even. In this case all even powers of R and all flips about a line through two sides form a subgroup H of D_n which acts transitively on the n-gon:

$$H = \{ I, R^2, R^4, \dots, R^{n-2}, F_1, F_2, \dots, F_m \}.$$

One needs to check that H is a subgroup, that is, H is closed under products. Exercise for the reader.

Now finally suppose that the orbit is three dimensional. Then the orbit is a regular solid, that is, the orbit is a Platonic solid. There are five Platonic solids, the tetrahedron, cube, octahedron, dodecahedron and the icosahedron. Platonic solids have duals. Given a platonic solid put a vertex at the centre of every face to get another platonic solid. Note that the dual of the dual is the original Platonic solid. The tetrahedron is self-dual, the dual of the cube is the octahedron and the dual of the dodecahedron is the icosahedron. It is clear that a Platonic solid and its dual have the same symmetry group.

If g fixes three independent vectors then g is the identity. Thus H is a subgroup of the symmetry group of the Platonic solid.

So we just need to find the symmetry groups of the tetrahedron, cube and dodecahedron and find all of their transitive subgroups.

The tetrahedron has four vertices. Thus the symmetry group is a subgroup of S_4. Pick two vertices. This defines an edge and reflection through the plane through the midpoint of this edge and the other two vertices switches these two vertices and fixes the other two vertices. This gives a transposition and so we get the whole of S_4.

Suppose we only consider transformations of determinant one. Half of the symmetries have determinant one and in fact they give a normal subgroup of S_4. Thus symmetries of determinant one give A_4.

In fact if we fix a vertex then we can rotate about an axis through the vertex through an angle of $2\pi/3$. This gives a 3-cycle and we can get any 3-cycle this way. The 3-cycles generate A_4. Thus the subgroup of all symmetries of determinant 1 is A_4. This acts transitively on the vertices of the tetrahedron.

Consider the possiblities for H. As H acts transitively on the vertices its order is divisible by 4. Thus H has order 4, 8, 12 or 24.

If H has order 24 it is S_4. If it has order 12 it is A_4. Suppose that H is a transitive subgroup of order 8. If every element of H has order at most 2 then H is abelian. The elements of order two are transpositions and products of transpositions. There are three products of transpositions. The remaining four elements must be transpositions and they must be disjoint; this is simply not possible. Thus H contains a 4-cycle. There are only three permutations which are the product of two disjoint transpositions. Thus H contains a transposition.

Suppose the 4-cycle is $(1,2,3,4)$. If H contains $(1,2)$ we get the whole of S_4, not possible. If H contains $(1,3)$ then H contains five elements of the symmetries of a square. Thus H contains all symmetries of the square, so that $H \simeq D_4$.

Finally suppose that H has order 4. If H contains a 4-cycle then it is the subgroup generated by this 4-cycle. H cannot contain a transposition, since the stabiliser of any vertex is trivial. The only possibility is that

$$H = V = \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\} \simeq \mathbb{Z}_2 \times \mathbb{Z}_2.$$

In this case $H \subset A_4$.

The cube has eight vertices. Let us first consider the symmetries H_+ of determinant one.

Consider the stabiliser of a vertex. Then the opposite corner is fixed as well. We can rotate the cube through an angle of $2\pi/3$ about an axis through those two vertices. This gives us three symmetries. Thus the symmetry group of the cube has order $3 \times 8 = 24$. As a check consider the stabiliser of face. There are six faces. If we fix the top face then we have to fix the bottom face as well. We can then rotate about a vertical axis through the centre of this pair of opposite faces through an angle of $\pi/2$. Thus the symmetry group of the cube has order $6 \times 4 = 24$. We can also consider the edges. The cube has 12 edges. If we fix an edge then we can simply swivel through an angle of π through an axis through this edge and the diagonally opposite edge. This gives us $12 \cdot 2 = 24$ symmetries.

It is natural to guess that the symmetry group of a cube is S_4. After a little bit of experimentation, one can proceed as follows. Note that there are four diagonals, that connect opposite vertices of the cube. It is clear that the symmetries of the cube act transitively on the diagonals. Suppose that we fix a diagonal. Then we can rotate the cube through an angle of $2\pi/3$, as described above. This gives us three elements of the stabiliser of a diagonal. Then we can switch the two opposite vertices. This gives us at least 6 elements of the stabiliser. But then this gives us 24 symmetries. Thus H_+ is isomorphic to S_4, realised as the symmetries of the diagonals.

Note that the full symmetry group H of the cube contains reflection r in the origin. This switches all diagonal vertices. Observe that this symmetry lies in the centre. This is geometrically clear (relabelling the vertices won't change the fact that we are simply reflecting in the origin) and even clearer algebraically, since this symmetry is represented by the matric $-I_3$, and this matrix lies in the centre of $G = \mathrm{GL}(3, \mathbb{R})$.

This gives us two normal subgroups. $R \simeq \mathbb{Z}_2$ generated by r and $H_+ \subset H$ which has index two (or is the kernel of the determinant). $R \cap H_+$ is the trivial group and R commutes with H_+. Thus

$$H \simeq H_+ \times R \simeq S_4 \times \mathbb{Z}_2.$$

Consider transitive subgroups T. The cube has eight vertices and so any transitive subgroup must have 8, 16, 24 or 48 elements. If there are 48 elements then we get $T = H \simeq S_4 \times \mathbb{Z}_2$.

Suppose that T has index two. Consider $T_+ = T \cap H_+$. If $T_+ = H_+$ then $T = H_+ \simeq S_4$. Otherwise $T_+ \subset T \simeq S_4$ has index two. But then $T_+ \simeq A_4$.

Two subgroups of index 2 are S_4 and $A_4 \times \mathbb{Z}_2$.

Now consider the dodecahedron. As before consider H_+ the symmetries of the dodecahedron which have determinant one. This has 20 vertices. If one fixes a vertex then one has to fix the opposite vertex and we can rotate the dodecahedron through an angle of $2\pi/3$ about this axis. Thus the order of the symmetry group is $20 \cdot 3 = 60$. As a check there are 12 faces. If one fixes a face then one has to fix the opposite face. One can rotate through an angle of $2\pi/5$ about the line through the centre of opposite faces. This gives us $12 \cdot 5 = 60$ symmetries. Finally there are $5 \cdot 12/2 = 30$ edges. If one fixes an edge then one has to fix the opposite edge and all one can do is rotate through an angle of π about an axis through the centre of opposite edges. This gives us $30 \cdot 2 = 60$ symmetries.

It is natural to guess that H_+ is isomorphic to A_5. There are 30 edges. One can group these into five groups of six, as follows. Suppose that e is an edge. Pick a face to which e belongs. This gives a pentagon of which e is one side. Let f be the edge that does not belong to this pentagon that goes through the vertex opposite e.

Let $\sim$ be the equivalence relation generated by this relation: $e \sim e'$ if we can find $e_1, e_2, \dots, e_n$ such that $e = e_1$, e_i and e_{i+1} are related as described above and $e' = e_n$. The corresponding partition has five equal partitions of cardinality 6.

This gives us a representation

$$\rho \colon H_+ \longrightarrow S_5$$

It is not hard to check that symmetry that fixes the five parts of the partition must be the identity, so that the kernel is trivial. The image is a subgroup of index 2; it must be A_5.

Note that the full symmetry group contains R so that arguing as before we must have

$$H \simeq H_+ \times R \simeq A_5 \times \mathbb{Z}_2.$$

Definition 24.2. *Let G be a group.*

*We say that G has the **Jordan property** if there is an integer $k > 0$ with following property:*

If $H \subset G$ is any finite subgroup then there an abelian subgroup $A \subset H$ whose index is at most k.

Note that if the index is at most k then H is somehow close to an abelian group.

Theorem 24.3 (Jordan)**.** $\mathrm{GL}(n, \mathbb{R})$ *has the Jordan property.*

Let's check Jordan's result for $n = 1$, 2 and 3.

$$\mathrm{GL}(1, \mathbb{R}) \simeq \mathbb{R}^*,$$

the non-zero reals under multiplication. This is abelian and so we can take $k = 1$. In fact we already saw that the biggest finite subgroup is $\{\pm 1\}$.

Now suppose $n = 2$. If $H \subset \mathrm{GL}(2, \mathbb{R})$ is a finite subgroup then we already saw that $H \subset D_n$. It is pretty clear the worse case is when $H = D_n$.

In this case the group of rotations is an abelian subgroup of index 2. Thus $\mathrm{GL}(2, \mathbb{R})$ is Jordan and we can even take $k = 2$.

Now suppose $n = 3$. The only remaining cases are the symmetry groups of the Platonic solids. There are only finitely many of these, so we could just take the maximum order (120). But we can do much better than this.

The symmetry group of the tetrahedron is S_4. This is not abelian. A_4 is not abelian either. Nor is D_4. But V is abelian and this has index 6. So $k = 6$ works.

The symmetry group of the cube is $S_4 \times \mathbb{Z}_2$. The $\mathbb{Z}_2$ plays no role (exercise for the reader). So we are down to S_4, and we already saw that $k = 6$ works.

The symmetry group of the dodecahedron is $A_5 \times \mathbb{Z}_2$. The factor of $\mathbb{Z}_2$ plays no role, as before. A_5 is not abelian. It contains a group of order 5, which is abelian. The index is 12. One cannot do better than this.

Thus $k = 12$ is the optimal value.