22. Group actions

Even though one defines a group abstractly the only sensible way to think about a group is as a group of symmetries, or what comes to the same thing, as a group of permutations.

Group actions are to permutations as equivalence relations are to partitions. Even if we are really only interested in realising a group as a permutation group, group actions are much easier to manipulate, even though the data of a group action is the same as the data of a group homomorphism.

Definition 22.1. *Let S be a set and let G be a group.*

A ***group action*** *is a function*

$$G \times S \longrightarrow S \qquad \text{given as} \qquad (g, s) \longrightarrow g \cdot s$$

that satisfies

(1) *For every $s \in S$ we have*

$$e \cdot s = s.$$

(2) *For every $s \in S$ and g, $h \in G$.*

$$(gh) \cdot s = g \cdot (h \cdot s).$$

In words, the identity of G acts as the identity on S and to apply gh is the same as to first apply h and then to apply g.

Definition-Lemma 22.2. *Suppose the group G acts on the set S.*

Define an equivalence relation $\sim$ on S by the rule $a \sim b$ if and only if there is an element $g \in G$ such that $g \cdot a = b$.

Proof. We have to check that $\sim$ is reflexive, symmetric and transitive.

If $s \in S$ then $e \cdot s = s$ so that $s \sim s$ and $\sim$ is reflexive.

If s and $t \in S$ and $s \sim t$ then we may find $g \in G$ such that $g \cdot s = t$. In this case

$$\begin{aligned} g^{-1} \cdot t &= g^{-1} \cdot (g \cdot s) \\ &= (g^{-1}g) \cdot s \\ &= e \cdot s \\ &= s. \end{aligned}$$

Thus $t \sim s$ and $\sim$ is symmetric.

Now suppose $r \sim s$ and $s \sim t$. Then we may find g and $h \in G$ such that $g \cdot r = s$ and $h \cdot s = t$. In this case

$$\begin{aligned}(gh) \cdot r &= g \cdot (h \cdot r)\\ &= g \cdot s\\ &= t.\end{aligned}$$

Thus $r \sim t$ and $\sim$ is transitive. □

Note that in the course of the proof we saw that g^{-1} acts as the inverse of g.

Definition 22.3. *The equivalence classes of the equivalence relation above are called* ***orbits****.*

The action is called ***transitive*** *if there is one orbit.*

Proposition 22.4. *Let G be a group and let S be a set.*

The data of an action of G on S is the same as the data of a ***representation****, a group homomorphism*

$$\phi \colon G \longrightarrow A(S).$$

Proof. Suppose we are given an action of G in S. If we fix g then we get a function

$$\sigma \colon S \longrightarrow S \qquad \text{given by} \qquad \sigma(s) = g \cdot s.$$

It is easy to see that the inverse of σ is given by the action of g^{-1}. Thus $\sigma \in A(S)$ is a permutation of S. This gives us a function

$$\phi \colon G \longrightarrow A(S) \qquad \text{given by} \qquad \phi(g) = \sigma.$$

Suppose that g and $h \in G$ and let $\sigma = \phi(g)$, $\tau = \phi(h)$ and $\rho = \phi(gh)$. We check that

$$\rho = \tau\sigma.$$

Both sides are permutations of S and so it suffices to show they have the same effect on an element $s \in S$. We have

$$\begin{aligned}\rho(s) &= (gh) \cdot s\\ &= g \cdot (h \cdot s)\\ &= g \cdot (\sigma(s))\\ &= \tau(\sigma(s))\\ &= (\tau \circ \sigma)(s)\\ &= (\tau\sigma)(s).\end{aligned}$$

Thus ϕ is a group homomorphism and we get a representation.

Now suppose we are given a representation, a group homomorphism

$$\phi\colon G \longrightarrow A(S).$$

Define an action

$$G \times S \longrightarrow S \qquad \text{by the rule} \qquad g \cdot s = \phi(g)(s).$$

It is straightforward to check that we do get an action and going backwards and forwards from action to representation are inverses to each other. □

Example 22.5. *D_n acts on the vertices of a regular n-gon.*

The action is the obvious one and the action is transitive. The corresponding representation is the standard one.

There are two natural ways a group acts on itself.

Example 22.6. *Let G be a group.*

G acts on the set G by left translation

$$G \times G \longrightarrow G \qquad \text{given by} \qquad g \cdot s = gs.$$

The action is transitive. The corresponding representation is the one given by Cayley's theorem.

More generally, let H be a subgroup of G. Then G acts on the left cosets S of H in G in the obvious way

$$G \times S \longrightarrow S \qquad \text{given by} \qquad g \cdot (aH) = (ga)H.$$

The action is transitive.

Example 22.7. *Let G be a group.*

G acts on itself by conjugation

$$G \times G \longrightarrow G \qquad \text{given by} \qquad g \cdot s = gsg^{-1}.$$

Note that the orbits are precisely the conjugacy classes of G.

One key property of group actions is that it is easy to count the size of an orbit:

Definition-Lemma 22.8. *Suppose the group G acts on the set S. Suppose that $s \in S$.*

The ***stabiliser*** *of s, denoted* Stab(s), *is the subgroup*

$$H = \{\, g \in G \mid g \cdot s = s. \,\}$$

Let O be the orbit of s. Then

$$|O| = [G : H].$$

In words the cardinality of the orbit of s is simply the index of the stabiliser of s.

In particular the cardinality of an orbit divides the order of G.

Proof. We first check that H is a subgroup.

H is non-empty as $e \in H$. We check that H is closed under products and inverses.

Suppose that g and $h \in H$. We have

$$\begin{aligned}(gh) \cdot s &= g \cdot (h \cdot s)\\ &= g \cdot s\\ &= s.\end{aligned}$$

Thus $gh \in H$ and H is closed under products.

Now suppose that $g \in H$. Then $g \cdot s = s$ so that $g^{-1} \cdot s = s$. Thus $g^{-1} \in H$ and H is closed under inverses. Thus H is a subgroup.

Define a function

$$f : G \longrightarrow S \qquad \text{by the rule} \qquad f(g) = g \cdot s.$$

The image of f is the orbit O of s. Define a relation $\sim$ on G by the rule $a \sim b$ if and only if $f(a) = f(b)$.

Claim 22.9. *$a \sim b$ if and only if $a^{-1}b \in H$.*

Proof of (22.9).

$$\begin{aligned}a \sim b \qquad &\text{if and only if } f(a) = f(b)\\ &\text{if and only if } a \cdot s = b \cdot s\\ &\text{if and only if } a^{-1} \cdot (b \cdot s) = s\\ &\text{if and only if } (a^{-1}b) \cdot s = s\\ &\text{if and only if } a^{-1}b \in H.\end{aligned}$$

□

Note that the relation $a \sim b$ if and only if $a^{-1}b \in H$ is the relation used to define the left cosets of H in G. It follows that the inverse image of point of O is simply a left coset, which has cardinality the order of H. Thus the number of elements of G is precisely

$$|G| = |O| \cdot |H|.$$

Dividing by $|H|$ and using Lagrange we get

$$|O| = [G : H].$$

□

Example 22.10. *Suppose that $G = D_n$ and S is the set of vertices of a regular n-gon.*

Fix a vertex a. No rotation fixes a but there is one flip that fixes a (it is the flip that either goes through the opposite vertex, if n is even, or the opposite edge if n is odd). Thus the stabiliser H of a has two elements.

The action is transitive and S has n elements. On the other hand D_n has $2n$ elements, so that the index of H is also n, as expected.

We will need another easy result about group actions:

Lemma 22.11. *Suppose that G acts on the set S.*

If $g \cdot s = t$ then

$$\operatorname{Stab}(t) = g \operatorname{Stab}(s) g^{-1}.$$

Proof. We show that the RHS is contained in the LHS. Suppose that $h \in \operatorname{Stab}(s)$. We have

$$\begin{aligned}(ghg^{-1}) \cdot t &= (ghg^{-1}) \cdot (g \cdot s) \\ &= (gh) \cdot ((g^{-1}g) \cdot s) \\ &= (gh) \cdot (e \cdot s) \\ &= (gh) \cdot s \\ &= g \cdot (h \cdot s) \\ &= g \cdot s \\ &= t.\end{aligned}$$

Thus $ghg^{-1} \in \operatorname{Stab}(t)$ and it follows that

$$\operatorname{Stab}(t) \supset g \operatorname{Stab}(s) g^{-1}.$$

Now apply the same result to t and g^{-1} to get

$$\operatorname{Stab}(g^{-1} \cdot t) \supset g^{-1} \operatorname{Stab}(t) g.$$

Conjugating both sides by g and observing that $s = g^{-1} \cdot t$ gives

$$\operatorname{Stab}(t) \subset g \operatorname{Stab}(s) g^{-1}. \qquad \square$$

In words, the stabiliser of $g \cdot s$ is the conjugate of the stabiliser of s by g.