20. Groups of small order

It is the aim of this section to classify all groups of order at most ten, up to isomorphism. To do this we recall some basic results.

First note that for every natural number n, there is at least one group of order n, namely a cyclic group of order n.

Lemma 20.1. *Let G be a group of order a prime p.*

Then G is cyclic.

Proof. Pick any element g of G other than the identity and let H be the subgroup generated by g. Then the order of H is greater than one and divides the order of G, by Lagrange. As the order of G is a prime, it follows that the order of H is the order of G, so that $H = G$. Therefore G is cyclic, generated by any element other than the identity. $\square$

Look at the numbers from one to ten. Of these, 2, 3, 5 and 7 are prime. Thus by (20.1) there is exactly one group of order 1, 2, 3, 5 and 7, up to isomorphism.

The numbers that are left are 4, 6, 8, 9 and 10. The next thing to do is to start looking for interesting subgroups. The easiest way to find a subgroup is to pick an element and look at the cyclic subgroup that it generates.

Lemma 20.2. *Let G be a group in which every element has order at most two.*

Then G is abelian.

Proof. Suppose that a, b and ab all have order at most two. We will show that a and b commute. By assumption

$$\begin{aligned} e &= (ab)^2 \\ &= abab. \end{aligned}$$

As a and b are their own inverses, multiplying on the left by a and then b, we get

$$ba = ab. \qquad \square$$

On the other hand, the classification of finite abelian groups is easy. There are two of order 4,

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \qquad \text{and} \qquad \mathbb{Z}_4,$$

one of order six,

$$\mathbb{Z}_6,$$

three of order 8,

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \qquad \mathbb{Z}_2 \times \mathbb{Z}_4 \qquad \text{and} \qquad \mathbb{Z}_8,$$

two of order nine,

$$\mathbb{Z}_3 \times \mathbb{Z}_3 \qquad \text{and} \qquad \mathbb{Z}_9,$$

and one of order ten

$$\mathbb{Z}_{10}.$$

Let us start with order four. Let $g \in G$ be an element other than the identity. Then the order of g is 2 or 4. If it is four then G is cyclic. Otherwise g has order two. If G is not cyclic then every element, other than the identity, must have order two, and G is abelian, by (20.2). Thus every group of order 4 is abelian.

Now suppose that G has order six. If G is abelian then G is cyclic. Otherwise, every element of G has order two or three. By (20.2) not every element has order at most two. Let a be an element of order three. Let $H = \langle a \rangle$.

Lemma 20.3. *Let G be a group and let H be a subgroup of index two. Then H is normal in G.*

Proof. It suffices to prove that the set of left cosets is equal to the set of right cosets.

The left cosets partition the elements of G into two parts. One part is equal to H and, by definition of a partition, the other part is the complement of H. By the same token, the right cosets consist of H and its complement.

Hence both partitions are equal and H is normal. □

Pick $b \in G$, where $b \notin H$. As H has index two, G/H has order two. Thus $b^2 \in H$. If $b^2 \neq e$, then $b^2 = a$ or $b^2 = a^2$ and b has order six, a contradiction. Thus $b^2 = e$ and b has order two. Clearly $G = \langle a, b \rangle$. Consider the conjugate of a by b,

$$bab^{-1}.$$

As H is normal in G, $bab^{-1} \in H$, so that $bab^{-1} = a$ or $bab^{-1} = a^2$. If the former then $ab = ba$ and G is abelian. Otherwise G is isomorphic to D_3, as they both have the same presentation. Thus there are two groups of order 6, a cyclic group and S_3.

Now suppose that the order is ten. If G is not abelian, then every element, other than the identity must have order 2 or 5. Not every element has order two. Let a be an element of order five. Let $H = \langle a \rangle$. Then H has index two. Thus H is normal in G. Let $b \in G$, $b \notin H$. As before $b^2 = e$. Once again consider the conjugate of a by b,

$$bab^{-1}$$

This is an element of H of order five. Thus $bab^{-1} = a^i$, some $i \neq 0$. We may suppose that $i \neq 1$, else G is abelian. If $i = 4$, then $bab^{-1} = a^{-1}$ and G is isomorphic to D_5, the symmetries of a regular pentagon.

Suppose that $bab^{-1} = a^2$. Then

$$\begin{aligned} a &= b^2ab^{-2} \\ &= b(bab^{-1})b^{-1} \\ &= ba^2b^{-1} \\ &= (bab^{-1})(bab^{-1}) \\ &= a^2a^2 \\ &= a^4. \end{aligned}$$

But then $a^4 = a$ and so $a^3 = e$, a contradiction. Similarly $bab^{-1} \neq a^3$. Thus a group of order ten is either cyclic or isomorphic to D_5.

Now suppose that G is a non-abelian group of order eight. There are no elements of order eight, as G is not cyclic and not every element has order two, by (20.2).

Thus G has an element a of order 4. Let $H = \langle a \rangle$. Then H has index two in G. Pick $b \in G$, with $b \notin H$. Then $b^2 \in H$. $b^2 \neq a$, a^3, otherwise b has order 8.

There are two possibilities, $b^2 = e$ or $b^2 = a^2$. Suppose first that $b^2 = e$. In this case consider, as before, the conjugate of a by b. As before, we must have $bab^{-1} = a^3$ and we have the dihedral group D_4. Call this group G_1.

Otherwise $b^2 = a^2$. Call this group G_2. Again we consider the conjugate of a by b. It must be a^3 as before. Note that this rule translates to $ba = a^3b$. Let $H = \langle a \rangle$ and $K = \langle b \rangle$. Then $G = \langle a, b \rangle = H \vee K = HK$, where we use the rule

$$ba = a^3b,$$

to prove that HK is closed under products and inverses, so that HK is a subgroup of G. We will see later that there is indeed a group of order eight with this presentation. Note that G_1 and G_2 are not isomorphic. Indeed G_1 has only two elements of order 4, a and a^3, whilst G_2 has at least three, a, a^3 and b.

Finally consider the case where G has order nine. Suppose that G is not abelian. Then every element of G, other than the identity must have order 3. Pick an element $a \neq e$ and let $H = \langle a \rangle$. Let S be the set of left cosets of H in G. Then S has three elements. As in the proof of Cayley's Theorem there is a group homomorphism

$$\phi \colon G \longrightarrow A(S) \simeq S_3$$

We send $g \in G$ to the permutation of S that sends aH to gaH. The kernel of ϕ is a normal subgroup of G that is contained in H. The image of ϕ has order at most six, and as G has order nine, the kernel of ϕ cannot be the trivial subgroup. It follows that $\operatorname{Ker}\phi = H$ so that H is normal in G.

Pick $b \in G - H$. Then bH is an element of G/H and so it must have order three. In particular $b^3 \in H$. But then $b^3 = e$, else b has order nine. Let $K = \langle b \rangle$. By symmetry K is normal in G. As $H \cap K = \{e\}$, it follows that the elements of H and K commute. But $G = \langle a, b \rangle$. Thus G is abelian, a contradiction.