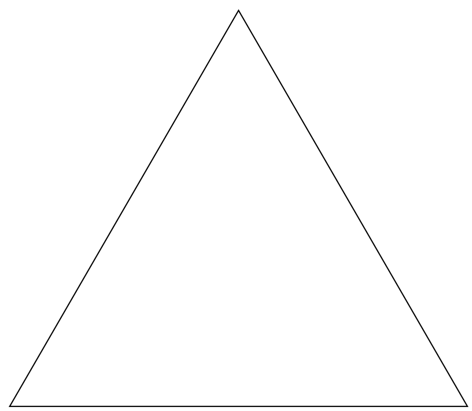
2. Examples of Symmetry groups

Suppose that we take an equilateral triangle and look at its symmetry group.

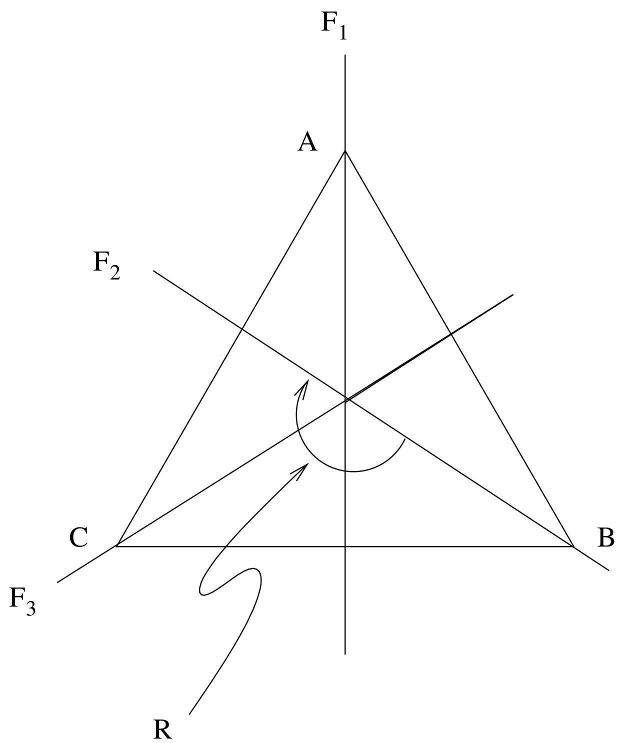


There are two obvious sets of symmetries. First one can rotate the triangle through 120°. Suppose that we choose clockwise as the positive direction and denote rotation through 120° as R. It is natural to represent rotation through 240° as R^2, where we think of R^2 as the effect of applying R twice.

If we apply R three times, represented by R^3, we would be back where we started. In other words we ought to include the trivial symmetry I, as a symmetry of the triangle (in just the same way that we think of zero as being a number). Note that the symmetry rotation through 120° anticlockwise, could be represented as R^{-1}. Of course this is the same as rotation through 240° clockwise, so that $R^{-1} = R^2$.

The other obvious sets of symmetries are flips. For example one can draw a vertical line through the top corner and flip about this line. Call this operation $F = F_1$. Note that $F^2 = I$, representing the fact that flipping twice does nothing.

There are two other axes to flip about, corresponding to the fact that there are three corners. Putting all this together we have



The set of symmetries we have created so far is then equal to

$$\{\, I, R, R^2, F_1, F_2, F_3 \,\}.$$

Is this all? The answer is yes, and it is easy to see this, once one notices the following fact; any symmetry is determined by its action on the vertices of the triangle. In fact a triangle is determined by its vertices, so this is clear. Label the vertices A, B and C, where A starts at the top, B is the bottom right, and C is the bottom left.

Now in total there are at most six different permutations of the letters A, B and C. We have already given six different symmetries, so we must in fact have exhausted the list of symmetries.

Note that given any two symmetries, we can always consider what happens when we apply first one symmetry and then another. However note that the notation RF is ambiguous. Should we apply R first and then F or F first and then R? We will adopt the convention that RF means first apply F and then apply R.

Now RF is a symmetry of the triangle and we have listed all of them. Which one is it? Well the action of RF on the vertices will take

$$\begin{aligned} A &\longrightarrow A \longrightarrow B \\ B &\longrightarrow C \longrightarrow A \\ C &\longrightarrow B \longrightarrow C. \end{aligned}$$

In total then A is sent to B, B is sent to A and C is sent to C. As this symmetry fixes one of the vertices, it must be a flip. In fact it is equal to F_3.

Let us now compute the symmetry FR. Well the action on the vertices is as follows

$$\begin{aligned} A &\longrightarrow B \longrightarrow C \\ B &\longrightarrow C \longrightarrow B \\ C &\longrightarrow A \longrightarrow A. \end{aligned}$$

So in total the action on the vertices is given as A goes to C, B goes to B and C goes to A. Again this symmetry fixes the vertex B and so it is equal to F_2.

Thus $RF = F_3 \neq F_2 = FR$.