17. The Alternating Groups

Consider the group S_3. Then this group contains a normal subgroup, generated by a 3-cycle.

Now the elements of S_3 come in three types. The identity, the product of zero transpositions; the transpositions, the product of one transposition, and the three cycles, products of two transpositions. Then the normal subgroup above, consists of all permutations that can be represented as a product of an even number of transpositions.

In general there is no canonical way to represent a permutation as a product of transpositions. But we might hope that the pattern above continues to hold in every permutation group.

Definition 17.1. *Let $\sigma \in S_n$ be a permutation.*

*We say that σ is **even** if it can be represented as a product of an even number of transpositions. We say that σ is **odd** if it can be represented as a product of an odd number of transpositions.*

The following result is much trickier to prove than it looks.

Lemma 17.2. *Let $\sigma \in S_n$ be a permutation.*

Then σ is not both an even and an odd permutation.

There is no entirely satisfactory proof of (17.2). Here is perhaps the simplest.

Definition 17.3. *Let $x_1, x_2, \dots, x_n$ be indeterminates and set*

$$f(x_1, x_2, \dots, x_n) = \prod_{i<j} (x_i - x_j).$$

For example, if $n = 3$, then

$$f(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3).$$

Definition 17.4. *Given a permutation $\sigma \in S_n$, let*

$$g = \sigma^*(f) = \prod_{i<j} (x_{\sigma(i)} - x_{\sigma(j)}).$$

Suppose that $\sigma = (1, 2) \in S_3$. Then

$$g = \sigma^*(f) = (x_2 - x_1)(x_2 - x_3)(x_1 - x_3) = -(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = -f.$$

The following Lemma is the key part of the proof of (17.2).

Lemma 17.5. *Let σ and τ be two permutations and let $\rho = \sigma\tau$. Then*

(1) $\sigma^*(f) = \pm f$.
(2) $\rho^*(f) = \sigma^*(\tau^*(f))$.
(3) $\sigma^*(f) = -f$, whenever σ is a transposition.

Proof. g is clearly a product of terms of the form $x_i - x_j$ or $x_j - x_i$. Thus $g = \pm f$. Hence (1).

$$\begin{aligned}\sigma^*(\tau^*(f)) &= \sigma^*(\prod_{i<j}(x_{\tau(i)} - x_{\tau(j)})) \\ &= \prod_{i<j}(x_{(\sigma(\tau(i))} - x_{\sigma(\tau(j))}) \\ &= \prod_{i<j}(x_{\rho(i)} - x_{\rho(j)}) \\ &= \rho^*(f).\end{aligned}$$

Hence (2).

Suppose that $\sigma = (a, b)$, where $a < b$. Consider the effect of applying σ to $x_i - x_j$, where $i < j$. Then the only terms of f affected by σ are the ones that involve either x_a or x_b.

Let

$$I = \{a, b\} \cap \{i, j\}.$$

If I is empty then $x_i - x_j$ is unaffected by σ.

Suppose that

$$I = \{b\}.$$

We consider the position of i. If $i < a$ then

$$x_i - x_j = x_i - x_b \qquad \text{is sent to} \qquad x_i - x_a$$

and there is no change in sign. If $a < i < b$ then

$$x_i - x_j = x_i - x_b \qquad \text{is sent to} \qquad x_i - x_a = -(x_a - x_i)$$

and there is a change in sign.

If $i = b$ then

$$x_i - x_j = x_b - x_j \qquad \text{is sent to} \qquad x_a - x_j$$

and there is no change in sign. The case $i > b$ is not possible. Now suppose that

$$I = \{a\}.$$

There are again three cases, depending on the position of j. If $j > b$ or $j = a$ there is no change in sign and if $a < j < b$ there is a change in sign. Thus the sign changes when I has one element cancel each other out.

Finally if

$$I = \{a, b\}$$

then

$$x_i - x_j = x_a - x_b \qquad \text{is sent to} \qquad x_b - x_a = -(x_a - x_b)$$

and there is a change in sign. In total then the sign changes. Hence (3). □

Proof. Suppose that σ is a product of an even number of transpositions. Then by (2) and (3) of (17.5), $\sigma^*(f) = f$. Similarly if $\sigma^*(f)$ is a product of an odd number of transpositions, then $\sigma^*(f) = -f$. Thus σ cannot be both even and odd. □

Definition-Lemma 17.6. *There is a surjective homomorphism*

$$\phi\colon S_n \longrightarrow \mathbb{Z}_2$$

The kernel consists of the even transpositions, and is called the ***alternating group*** A_n.

Proof. The map sends an even transposition to 0 and an odd transposition to 1. (2) of (17.5) implies that this map is a homomorphism. □

Note that half of the elements of S_n are even, so that the alternating group A_n has order $\frac{n!}{2}$. One of the most important properties of the alternating group is,

Theorem 17.7. *Suppose that* $n \geq 5$.

The only normal subgroup of S_n *is* A_n. *Moreover* A_n *is simple, that is,* A_n *has no proper normal subgroups.*

Recall that $V = \{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ is a normal subgroup of S_4. It is also therefore a normal subgroup of A_4.