16. Characteristic subgroups and Products

Recall that a subgroup is normal if it is invariant under conjugation. Now conjugation is just a special case of an automorphism of G.

Definition 16.1. *Let G be a group and let H be a subgroup. We say that H is a* ***characteristic subgroup*** *of G, if for every automorphism ϕ of G, $\phi(H) \subset H$.*

First an easy observation.

Lemma 16.2. *Let H be a characteristically normal subgroup of G.*

(1) H is normal in G.

(2) If ϕ is an automorphism of G then $\phi(H) = H$.

Proof. If $a \in G$ then let

$$\phi \colon G \longrightarrow G \qquad \text{given by} \qquad g \longrightarrow aga^{-1}$$

Then ϕ is an automorphism of G and

$$aHa^{-1} = \phi(H) \subset H.$$

Thus H is normal in G. This is (1).

Let ψ be the inverse of ϕ. Then ψ is an automorphism of G and so

$$\psi(H) \subset H.$$

Applying ϕ it follows that

$$\begin{aligned} H &= \phi(\psi(H)) \\ &\subset \phi(H). \end{aligned}$$

This gives (2). □

It turns out that most of the *general* normal subgroups that we have defined so far are all in fact characteristic subgroups.

Lemma 16.3. *Let G be a group and let $Z = Z(G)$ be the centre. Then Z is characteristically normal.*

Proof. Let ϕ be an automorphism of G. We have to show $\phi(Z) \subset Z$. Pick $z \in Z$. Then z commutes with every element of G. Pick an element x of G. As ϕ is a bijection, $x = \phi(y)$, for some $y \in G$.

We have

$$\begin{aligned} x\phi(z) &= \phi(y)\phi(z) \\ &= \phi(yz) \\ &= \phi(zy) \\ &= \phi(z)\phi(y) \\ &= \phi(z)x. \end{aligned}$$

As x is arbitrary, it follows that $\phi(z)$ commutes with every element of G. But then $\phi(z) \in Z$. Thus $\phi(Z) \subset Z$. □

Definition 16.4. *Let G be a group and let x and y be two elements of G. $x^{-1}y^{-1}xy$ is called the commutator of x and y.*

*The **commutator subgroup** of G is the group generated by all of the commutators.*

Lemma 16.5. *Let G be a group and let H be the commutator subgroup.*

Then H is characteristically normal in G and the quotient group G/H is abelian. Moreover this quotient is universal amongst all homomorphisms to abelian groups in the following sense.

Suppose that $\phi\colon G \longrightarrow G'$ is any homomorphism of groups, where G' is abelian. Then there is a unique homomorphism $G/H \longrightarrow G'$.

Proof. Suppose that ϕ is an automorphism of G and let x and y be two elements of G. Then

$$\phi(x^{-1}y^{-1}xy) = \phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y).$$

The last expression is clearly the commutator of $\phi(x)$ and $\phi(y)$. Thus $\phi(H) \subset H$ and so H is characteristically normal in G.

Suppose that aH and bH are two left cosets. Then

$$\begin{aligned}(bH)(aH) &= baH \\ &= ba(a^{-1}b^{-1}ab)H \\ &= abH \\ &= (aH)(bH).\end{aligned}$$

Thus G/H is abelian. Suppose that $\phi\colon G \longrightarrow G'$ is a homomorphism, and that G' is abelian. By the universal property of a quotient, it suffices to prove that the kernel of ϕ must contain H.

Since H is generated by the commutators, it suffices to prove that any commutator must lie in the kernel of ϕ. Suppose that x and y are in G.

Then $\phi(x)\phi(y) = \phi(y)\phi(x)$. It follows that

$$\phi(x)^{-1}\phi(y)^{-1}\phi(x)\phi(y) = \phi(x^{-1}y^{-1}xy)$$

is the identity in G'. Thus $x^{-1}y^{-1}xy$ is sent to the identity, that is, the commutator of x and y lies in the kernel of ϕ. □

Definition-Lemma 16.6. *Let G and H be any two groups.*

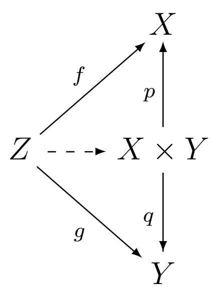
*The **product** of G and H, denoted $G \times H$, is the group, whose elements are the ordinary elements of the cartesian product of G and H as sets, with multiplication defined as*

$$(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2).$$

Proof. We need to check that with this law of multiplication, $G \times H$ becomes a group. This is left as an exercise for the reader. $\square$

Definition 16.7. *Let $\mathcal{C}$ be a category and let X and Y be two objects of $\mathcal{C}$. The* ***categorical product*** *of X and Y, denoted $X \times Y$, is an object together with two morphisms $p\colon X \times Y \longrightarrow X$ and $q\colon X \times Y \longrightarrow Y$ that are universal amongst all such morphisms, in the following sense.*

Suppose that there are morphisms $f\colon Z \longrightarrow X$ and $g\colon Z \longrightarrow Y$. Then there is a unique morphism $Z \longrightarrow X \times Y$ which makes the following diagram commute,



Note that, by the universal property of a categorical product, in any category, the product is unique, up to unique isomorphism. The proof proceeds exactly as in the proof of the uniqueness of a categorical quotient and is left as an exercise for the reader.

Lemma 16.8. *The product of groups is a categorical product.*

That is, given two groups G and H, the group $G \times H$ defined in (16.6) *satisfies the universal property of* (16.7).

Proof. First of all note that the two ordinary projection maps $p\colon G \times H \longrightarrow G$ and $q\colon G \times H \longrightarrow H$ are both homomorphisms (easy exercise left for the reader).

Suppose that we are given a group K and two homomorphisms $f\colon K \longrightarrow G$ and $g\colon K \longrightarrow H$. We define a map $u\colon K \longrightarrow G \times H$ by sending k to $(f(k), g(k))$.

It is left as an exercise for the reader to prove that this map is a homomorphism and that it is the only such map, for which the diagram commutes. $\square$