

MODEL ANSWERS TO THE FIFTH HOMEWORK

1. We proved in class that if one blows up a point the log discrepancy is at least two.

Now let ν be an arbitrary valuation. Pick a birational morphism $\pi: T \rightarrow S$, a composition of smooth blow ups, which realises the centre of ν as one of the exceptional divisors. Suppose that π is the composition of k blow ups, and that $E_1, E_2, \dots, E_k$ are the exceptional divisors, in the order in which they appear. We may assume that one cannot realise the centre of ν as a divisor with fewer than k blow ups. In this case the centre of ν is E_k and we may assume that $k > 1$.

Let $f: S_1 \rightarrow S$ be the first blow up, and let $\pi_1: T \rightarrow S_1$ be the induced birational morphism. Then π_1 is a composition of $k - 1$ blow ups. By induction, we have

$$K_T + \sum_{i=2}^k E_i = \pi_1^* K_{S_1} + \sum b_i E_i,$$

where $b_i \geq 2$ with equality if and only if $i = 2$. On the other hand

$$K_{S_1} + F = f^* K_S + 2F,$$

where F is the exceptional divisor of the blow up f , so that the strict transform of F on T is E_1 . Thus

$$E_1 + \sum_{i=2}^k c_i E_i = \pi_1^* F,$$

where c_i are positive integers. It follows that

$$\begin{aligned} K_T + \sum_{i=1}^k E_i &= K_T + E_1 + \sum_{i=2}^k E_i \\ &= p_{1*}^*(K_{S_1} - F) + \sum (b_i + c_i) E_i \end{aligned}$$

2. Pick m such that $m(K_X + \Delta)$ is Cartier. Then the log discrepancy takes values in the discrete set $\mathbb{Z}\langle 1/m \rangle$ and the result is clear.

It is also interesting to see what happens when $K_X + \Delta$ is not $\mathbb{Q}$ -Cartier. Suppose that there is a valuation of irrational log discrepancy less than zero. I claim that the set of log discrepancies is then dense in the real numbers. As in the lectures, we may assume that $X = S$ is a smooth

surface and $\Delta = (1 + \epsilon)C$, where C is a smooth curve, and $\epsilon > 0$ is irrational.

Blowing up along the repeated intersection of the exceptional divisor and the strict transform of C , we can create a component of coefficient $n\epsilon$, as in the lectures. In other words, we may assume that the coefficient of C is $n\epsilon$, for any $n > 0$. Now consider blowing up along the exceptional divisor, but away from C . After one blow up the coefficient is $n\epsilon - 1$. After m such blow ups the coefficient is $n\epsilon - m$. But the set

$$\{n\epsilon - m \mid (n, m) \in \mathbb{N}^2\},$$

is dense in $\mathbb{R}$, for any positive irrational number ϵ .

3. (i) As C_2 is the union of the two axes, the log canonical threshold is one.

(ii) We have to write down a log resolution. Let $\pi: Y \rightarrow X$ first blow up the singular point, then the intersection of the strict transform of C with the exceptional divisor and finally blow up the triple intersection of the strict transform of C , the strict transform of the old exceptional divisor and the new exceptional divisor. Label the exceptional divisor E_1 , E_2 and E_3 , in the order they appear. Then C has multiplicity 2, 3 and 6 along E_1 , E_2 and E_3 respectively. On the other hand, the log discrepancy of E_1 is 2, or E_2 is 3 and of E_3 is 5. Thus

$$K_Y + \lambda D + E_1 + E_2 + E_3 = \pi^*(K_X + \lambda C) + (2 - 2\lambda)E_1 + (3 - 3\lambda)E_2 + (6 - 5\lambda)E_3.$$

So the largest value of λ , such that the log discrepancies $2(1 - \lambda)$, $3(1 - \lambda)$ and $(6 - 5\lambda)$ are all non-negative is $5/6$. The log canonical threshold is $5/6$.

(iii) In this case a log resolution is given by blowing up twice. The multiplicity of C along E_1 and E_2 is 2 and 4 and the log discrepancy is 2 and 3. Thus

$$K_Y + \lambda D + E_1 + E_2 = \pi^*(K_X + \lambda C) + (2 - 2\lambda)E_1 + (3 - 4\lambda)E_2.$$

The log canonical threshold is therefore $3/4$.

(iv) The log canonical threshold is $1/m + 1/n$. The easiest way to see this is to use weighted blowups (or toric geometry). In terms of toric geometry, suppose that we make the weighted blow up corresponding to inserting a vector of type (m, n) . Suppose that the exceptional divisor is E . Then

$$K_Y + \lambda D + E = \pi^*(K_X + \lambda C) + aE,$$

where a is a function of λ . Now E is a copy of $\mathbb{P}^1$, but the twist is that Y is singular along E (in other words, the trick is to go to a log resolution, focus on the component which computes the log canonical threshold, and contract all the other components; the resulting surface

Y is then singular along E). Now there are then two singular points along E , and again by general theory, these singular points are cyclic quotient singularities of index m and n . This implies that

$$(K_Y + E)|_E = K_E + \sum \frac{m-1}{m}p + \frac{n-1}{n}q,$$

where p and q are the points of E corresponding to the two cyclic quotient singularities. On the other hand, D intersects E transversally in one point. Now at the log canonical threshold, $a = 0$ (indeed, $K_Y + E + \lambda D$ is log canonical). Thus

$$0 = (K_Y + E + \lambda D) \cdot E = -2 + \frac{m-1}{m} + \frac{n-1}{n} + \lambda.$$

Thus

$$\lambda = \frac{1}{m} + \frac{1}{n},$$

as conjectured.

4. Let $\pi: S \rightarrow Z$ contract a real K_S -extremal ray. As in the classical case, there are three cases, given by the dimension of Z .

Suppose that Z is a point. Then S is either a copy of $\mathbb{P}^2$, defined over the reals, or a copy of $\mathbb{P}^1 \times \mathbb{P}^1$, on which complex conjugation acts by switching the two fibrations.

Suppose that Z is a curve. Then Z is a curve over the reals. If p is a point of Z which is not equal to its complex conjugate, then the fibre over p is a copy of $\mathbb{P}^1$, with no real points. Otherwise if p is a real point, then there are two possibilities for the fibre. In the first, the fibre is a curve of genus zero over the reals (there are two such, those with real points $\mathbb{P}_{\mathbb{R}}^1$, and those with none, $x^2 + y^2 + z^2 = 0 \subset \mathbb{P}^2$). In the second the fibre is a union of two $\mathbb{P}^1$'s, joined at a real point, and complex conjugations switches the two copies. There is no limit to the number of these reducible fibres.

Finally suppose that Z is a surface. In this case, the exceptional locus is either irreducible, a curve of genus zero over the reals, or reducible, two complex conjugate -1 -curves.