

6. LOG RESOLUTIONS AND BIRATIONAL INVARIANTS

Definition 6.1. Let X be a normal projective variety and let D be a $\mathbb{Q}$ -divisor. The **Kodaira dimension** of D is equal to the maximum dimension of the image of X under the linear systems $|mkD|$,

$$\kappa(X, D) = \max_{m \in \mathbb{N}} \dim \phi_m(X),$$

where $\phi_m = \phi_{|mkD|}: X \longrightarrow \mathbb{P}^N$. Equivalently,

$$\kappa(X, D) = \limsup_{m \in \mathbb{N}} \frac{\log(h^0(X, \mathcal{O}_X(mD)))}{\log m}.$$

The **Kodaira dimension** of X is the Kodaira dimension of the canonical divisor, $\kappa(X) = \kappa(X, K_X)$.

The **plurigenera** of X are the dimensions of the space of global n -forms of weight m ,

$$P_m(X) = h^0(X, \mathcal{O}_X(mK_X)).$$

Just to be confusing $p_g(X) = P_1(X)$, the number of independent global n -forms.

The **irregularity** of X is the dimension of the space of 1-forms,

$$q(X) = h^0(X, \Omega_X^1).$$

Theorem 6.2. If X and Y are two smooth projective varieties which are birational then X and Y have the same invariants defined in (6.1).

Definition 6.3. Let X be normal variety and let $\mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf on X . We say that $\mathcal{I}$ is **principal** if X is smooth and every point of X has a neighbourhood with coordinates $x_1, x_2, \dots, x_n$ so that $\mathcal{I}$ is locally given by a single monomial.

We have the following celebrated result of Hironaka:

Theorem 6.4 (Principalisation of Ideals). Let M be a smooth variety and let $\mathcal{I}$ be an ideal sheaf on X .

Then there is a composition of smooth blow ups $\pi: Y \longrightarrow X$ along smooth centres, with support contained in the support of $\mathcal{O}_X/\mathcal{I}$, such that $\pi^*\mathcal{I}$ is a principal ideal.

Definition 6.5. Let (X, Δ) be a log pair.

We say that (X, Δ) is **log smooth**, if the pair (X, D) has global normal crossings (that is, every irreducible component of D is smooth and locally (in the analytic or étale topology) about any point of X , $(X, D = \sum \Delta_i)$ is isomorphic to $(\mathbb{C}^n, H_1 + H_2 + \dots + H_k)$ where $H_1, H_2, \dots, H_n$ are the coordinate hyperplanes).

A **log resolution** of (X, Δ) is a birational morphism $\pi: Y \rightarrow X$ such that $(Y, \Gamma = f_*^{-1}\Delta + E)$ is log smooth, where $f_*^{-1}\Delta$ is the strict transform of Δ and E is the sum of the exceptional divisors, and there is a divisor F , supported on the exceptional locus, such that F is π -ample.

Remark 6.6. Note that in the definition of a log resolution, we make no requirement that the locus where π is not an isomorphism is concentrated over any special locus in X (such as where X is singular).

One can restate the condition that $\mathcal{I}$ is principal by saying there is a divisor D such that

$$\mathcal{I} = \mathcal{O}_X(-D)$$

where (X, D) is log smooth.

Corollary 6.7. Every log pair (X, Δ) has a log resolution.

Proof. Embed $X \subset M$ inside a smooth variety, where X has codimension at least two. Let $\pi: N \rightarrow M$ be a birational morphism which principalises $\mathcal{I}_X \subset \mathcal{O}_M$. In particular the inverse image E of X is a divisor.

Equivalently, E is an exceptional divisor of π . Suppose that π is the composition of $\pi_i: M_{i+1} \rightarrow M_i$, where $M = M_0$ and $M = M_n$. Let i be the smallest index such that the inverse image E_i of X is a divisor on M_i . Then E_i must be the exceptional divisor of $\pi_{i-1}: M_i \rightarrow M_{i-1}$. Note that for smaller indices the centre of the blow up must intersect the inverse image of X in a proper subset. In this case the inverse image is the same as the strict transform $X_j \subset M_j$, $j < i$. Finally note the centre of each blow up must be contained in X_j , $j < i$.

It follows that the centre of the blow up π_i is $X_{i-1} \subset M_{i-1}$. But since we only blow up smooth centre, in fact X_{i-1} must be smooth. The induced birational morphism $X_{i-1} \rightarrow X$ is a resolution of singularities of X . So replacing X by its resolution, and Δ by the strict transform of Δ plus the exceptional locus, we may assume that X is log smooth. Now apply (6.4) to $\mathcal{I}_D \subset \mathcal{O}_X$. $\square$

Definition 6.8. Let X be a variety. We say that X is

- **rational** if X is normal and birational to $\mathbb{P}^n$, some n .
- **stably rational** if $X \times \mathbb{P}^n$ is rational, some n .
- **unirational** if X is normal and there is a dominant rational map $\phi: \mathbb{P}^n \dashrightarrow X$, some n .
- **rationally connected** if for every two points x and y there is a morphism $f: \mathbb{P}^1 \rightarrow X$ such that $f(0) = x$ and $f(\infty) = y$.

- **rationally chain connected** if for every two points x and y there are morphisms $f_i: \mathbb{P}^1 \rightarrow X$, $0 \leq i \leq k$, such that $f_0(0) = x$ and $f_i(\infty) = f_{i+1}(0)$, $0 \leq i \leq k-1$ and $f_k(\infty) = y$.
- **uniruled** if for every point $x \in X$ there is a non-constant morphism $f: \mathbb{P}^1 \rightarrow X$ such that $f(0) = x$.

Remark 6.9. *It is clear that rational implies stably rational, stably rational implies unirational, unirational implies rationally connected (indeed given x and $y \in \mathbb{P}^n$, the line connecting them shows that $\mathbb{P}^n$ is rationally connected and the image of a rationally connected variety is rationally connected), rationally connected implies rationally chain connected and rationally chain connected implies uniruled.*

Note that X is uniruled if and only if there is a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$. If $X = \mathbb{P}^1 \times C$, where C is a curve of genus at least one, then X is uniruled but not rationally chain connected. If $X = \mathbb{P}^1 \cup \mathbb{P}^1$, where the two copies of $\mathbb{P}^1$ are joined at a point, then X is rationally chain connected but not rationally connected. More generally, if X is rationally connected then it is irreducible. Let S be the cone over an elliptic curve. Then S is rationally chain connected but not rationally connected. Let $\pi: T \rightarrow S$ blow up the vertex. Then T is not rationally chain connected. On the other hand rationally connected is a birational invariant and to check that X is rationally connected it suffices to show that any two points belonging to an open subset can be connected by the image of $\mathbb{P}^1$. It is conjectured that rationally connected does not imply unirational. In fact every smooth quartic $X \subset \mathbb{P}^4$ is rationally connected but it is conjectured that a general smooth quartic is not unirational. Finally it is known that there are unirational varieties which are not rational (in fact every smooth cubic in $\mathbb{P}^4$ is unirational; the general one is not unirational. Further every smooth quartic in $\mathbb{P}^4$ is not rational and some of these are unirational.) We also know examples that show stably rational does not imply rational and unirational does not imply stably rational.

Lemma 6.10. *If X is uniruled then X is covered by rational curves C , such that $K_X \cdot C < 0$.*

In particular $\kappa(X) = -\infty$.

Proof. Let $\text{Hom}(\mathbb{P}^1, X)$ denote the moduli space of morphisms from $\mathbb{P}^1$ to X . Then there is a natural evaluation map,

$$\text{Hom}(\mathbb{P}^1, X) \times \mathbb{P}^1 \rightarrow X,$$

which sends the pair (f, t) to the point $f(t) \in X$. By assumption there is a component B of $\text{Hom}(\mathbb{P}^1, X)$, which parametrises non-constant

morphisms, for which this map is dominant. In characteristic zero, the differential of the map $\alpha: B \times \mathbb{P}^1 \longrightarrow X$ is surjective at some point (f, t) of B . On the other hand, there is a natural action of $\text{Aut}(\mathbb{P}^1)$ on $\text{Hom}(\mathbb{P}^1, X)$ and on B . It follows that the differential of α is surjective on the whole orbit of $\text{Aut}(\mathbb{P}^1)$. In particular the differential of α is surjective on the whole fibre $\mathbb{P}^1$ (since the action of $\text{Aut}(\mathbb{P}^1)$ is transitive). But then there is an open subset U of B for which the differential of α is surjective on $U \times \mathbb{P}^1$. Cutting by hyperplanes, we may assume that α is finite, in which case the differential is then an isomorphism on the whole fibre over the open set U . Then

$$K_X \cdot_f C \leq K_C = K_{\mathbb{P}^1} = -2.$$

Suppose to the contrary that $B \in |mK_X|$. Pick $C \not\subset B$. Then

$$0 \leq B \cdot C = (mK_X) \cdot C < 0,$$

a contradiction. □

Let $X = \mathbb{C}^3$ and let X_1 be the blow up of the origin of X . The exceptional divisor is then a copy of $\mathbb{P}^2$. Let X_2 be the blow up of X_1 along an smooth cubic in the exceptional divisor. Then the exceptional locus is a copy E of $\mathbb{P}^2$ joined to a $\mathbb{P}^1$ -bundle F over an elliptic curve, joined along a section and a cubic. Then $E \cup F$ is a rationally chain connected variety, and yet F is not rationally connected. To connect two points of F , f_1 and f_2 , let F_1 and F_2 be the two fibres which contain them. Now let x_1 and x_2 be the points in E which meets these two fibres. Let l be the line connecting x_1 to x_2 . Then $F_1 \cup l \cup F_2$ connects f_1 to f_2 .

Example 6.11. *Let*

$$X = C \times \mathbb{P}^2 \cup \mathbb{P}^2 \times C \subset \mathbb{P}^2 \times \mathbb{P}^2.$$

Then X is rationally chain connected, but neither component is rationally connected.

Remark 6.12. *If X is rationally connected then $q(X) = 0$, and in fact all symmetric powers of all p -forms vanish.*

Theorem 6.13 (Elimination of indeterminacy). *Let $\phi: X \dashrightarrow Y$ be a rational map between projective varieties.*

Then there are morphisms $p: W \longrightarrow X$ and $q: W \longrightarrow Y$, where p is a composition of smooth blow ups along smooth centres, W is smooth

and there is a commutative diagram

$$\begin{array}{ccc} & W & \\ p \swarrow & & \searrow q \\ X & \overset{\phi}{\dashrightarrow} & Y. \end{array}$$

Moreover if X is smooth and $y \in Y$ is in the image of the indeterminacy locus of ϕ , then there is a non-constant morphism $f: \mathbb{P}^1 \rightarrow Y$ such that $f(0) = y$.

Proof. Pick an embedding of $Y \subset \mathbb{P}^n$ into projective space and let H be a hyperplane section. Let $\phi^*H = M + F$ be the decomposition of ϕ^*H into its fixed and mobile parts. Then the linear system $|M|$ defines ϕ . Let B the scheme theoretic base locus of $|M|$. Then $B = 0$ if and only if ϕ is a morphism (or perhaps better, extends to a morphism). Note that the codimension of B is at least two.

Let $\mathcal{I}_B$ be the ideal sheaf of B . Let $p: X \rightarrow Y$ be the birational morphism, whose existence is guaranteed by (6.4). Let $q: W \dashrightarrow Y$ be the induced rational map. Then $q^*H = M_1 + F_1$ is the decomposition of q^*H into its mobile and fixed parts, where $F_1 = p^*B$ and $M_1 = p^*M - F_1$. But then $|M_1|$ is base point free, so that q is a morphism.

Let $V \subset X$ be the indeterminacy locus of ϕ , and let $Z = qp^{-1}(V)$. If $x \in V$, then the $qp^{-1}(x)$ is positive dimensional. Since the image of a rationally chain connected variety is rationally chain connected it suffices to prove that the fibres of p are rationally chain connected. We prove this by induction on the number of blow ups. Suppose that p factors as $p_1: W_1 \rightarrow X$ and $\pi: W \rightarrow W_1$, where π is a smooth blow up of $B \subset W_1$. By induction the fibres of p_1 are rationally chain connected. Let $E_1 \subset W$ be the intersection of the exceptional divisor E with a fibre of p and let $B_1 \subset W_1$ be the image of E_1 (so that B_1 is a fibre of p_1). Then the fibres of E_1 over B_1 are projective spaces, whence rationally connected. If f_1 and f_2 are two points of two fibres F_1 and F_2 , then pick x_1 and x_2 belonging to the fibres and the strict transform G of B_1 . Then we can find a rational curve connecting f_1 to x_1 in F_1 , a chain of rational curves connecting x_1 to x_2 in G and a rational curve connecting x_2 to f_2 in F_2 . The resulting chain connects f_1 to f_2 in the fibre E_1 . $\square$

To get some idea of how to prove (6.13), consider the case of smooth projective surfaces. To emphasize this point, we change notation and consider $\phi: S \dashrightarrow Y$, where S is a smooth projective surface. As M is mobile it is nef (here is one important place where we use the fact that S is a surface). We proceed by induction on $d = M^2 \geq 0$. Suppose

that ϕ is not defined at $x \in |M|$. Then x is a base point of $|M|$. Let $\pi: S_1 \rightarrow S$ be the blow up. Let $\phi_1: S_1 \rightarrow Y$ be the induced rational map. If ϕ_1 is given by M_1 , then

$$M_1 = \pi^*M - mE,$$

where $m > 0$ is a positive integer (in fact $\phi^*M = M_1 + mE$ gives the decomposition into fixed and mobile parts). Now

$$M_1^2 = (\pi^*M - mE)^2 = d - m^2 < d.$$

By induction ϕ_1 is a composition of smooth blow ups. As $\phi = \phi_1 \circ \pi$ it follows that ϕ is a composition of smooth blow ups.

Definition 6.14. *Let (X, Δ) be a log pair. If π is any birational morphism and $E_1, E_2, \dots, E_k$ are the exceptional divisors, then we may write*

$$K_Y + \Gamma = K_X + \pi_*^{-1}\Delta + E = \pi^*(K_X + \Delta) + \sum a_i E_i,$$

for rational numbers $a_1, a_2, \dots, a_k$, where $\pi_*^{-1}\Delta$ is the strict transform of Δ and $E = \sum E_i$ is the sum of the exceptional divisors. The number $a_i = a(E_i, X, \Delta)$ is called the **log discrepancy** of the divisor E_i .

The log discrepancy of (X, Δ) is the infimum of the log discrepancies over all exceptional divisors of all birational morphisms.

Note that it is not necessary to assume that $\Delta \geq 0$ to define the log discrepancy. We only need that X is normal and $K_X + \Delta$ is $\mathbb{Q}$ -Cartier.

Example 6.15. *Let $X = \mathbb{A}^2$ and let $\pi: Y \rightarrow X$ be the blow up of the origin with exceptional divisor E .*

What is the log discrepancy of E with respect to K_X (that is, with respect to $(X, 0)$) ? There is no boundary Δ and so we may write

$$K_Y + E = \pi^*K_X + aE,$$

where a is the log discrepancy. To determine a we dot both sides with respect to E and see what we get

$$\begin{aligned} -2 &= \deg K_{\mathbb{P}^1} \\ &= \deg K_E \\ &= (K_Y + E) \cdot E \\ &= (\pi^*K_X + aE) \cdot E \\ &= \pi^*K_X \cdot E + aE^2 \\ &= K_X \cdot \pi_*E - a \\ &= -a. \end{aligned}$$

Thus the log discrepancy is 2. One can check that this is the log discrepancy of X .

Now suppose that X is the cone over a rational normal curve of degree d . Let p be the vertex of the cone and let $\pi: Y \rightarrow X$ blow up p . Then the exceptional divisor E is again a copy of $\mathbb{P}^1$ but now $E^2 = -d$, a function of d . Following the same calculation the log discrepancy a of E satisfies the equation

$$-2 = a(-d) \quad \text{so that} \quad a = \frac{2}{d}.$$

One can check that this is the log discrepancy of X . Note that as d increases the log discrepancy decreases.

In general the expectation is that worse singularities have smaller log discrepancy.

Definition 6.16. Let K/k be a field extension. A **valuation** ν of K/k is a map

$$\nu: K \rightarrow \mathbb{Z} \cup \{\infty\},$$

such that

- (1) $\nu(f) = \infty$ if and only if $f = 0$.
- (2) $\nu(fg) = \nu(f) + \nu(g)$.
- (3) $\nu(f + g) \geq \max(\nu(f), \nu(g))$.
- (4) $\nu(k - \{0\}) = \{0\}$.

Example 6.17. Let X be a normal projective variety and let $D \subset X$ be a prime divisor. Then the order of vanishing of a rational function along D determines a valuation,

$$\nu_D(f) = \text{mult}_f D.$$

If ν is a valuation such that $\nu = \nu_E$ for some divisor E , possibly exceptional, then we will call ν a **divisorial valuation**. The **centre** of ν is the image of E in X .

The language of valuations provides a convenient way to refer to the same exceptional divisors, on different models. Note that if $E_1 \subset Y_1$ and $E_2 \subset Y_2$ are two divisors on birational varieties Y_1 and Y_2 , then $\nu_{E_1} = \nu_{E_2}$ if and only if there is a birational map $\phi: Y_1 \dashrightarrow Y_2$ which is an isomorphism in a neighbourhood of the generic points of E_1 and E_2 .

The log discrepancy is a birational invariant, in the following sense:

Lemma 6.18. Let (X, Δ) be a log pair and let ν be a valuation.

The log discrepancy only depends on ν .

Proof. Suppose that we are given $\pi_i: Y_i \rightarrow X$ two birational morphisms on which the centre of ν_i is a divisor E_i . If $\phi: Y_1 \dashrightarrow Y_2$ is the induced birational map then ϕ is an isomorphism at the generic point of E_1 . We may write

$$K_{Y_i} + \Gamma_i = \pi_i^*(K_X + \Delta) + a_i E_i + J_i,$$

where J_i does not involve E_i and we want to show that $a_1 = a_2$. Pick a meromorphic differential form ω_2 on Y_2 and let $\omega_1 = \phi^* \omega_2$. Then

$$a_i = 1 - \text{mult}_{E_i} \pi_i^*(K_X + \Delta) + \text{mult}_{E_i} \omega_i,$$

which is independent of i . $\square$

Definition 6.19. We say that a log pair (X, Δ) is **canonical** if the log discrepancy is at least one.

Lemma 6.20. Let $\phi: X \dashrightarrow Y$ be a birational map between two projective varieties with canonical singularities and let m be a positive integer.

Then there is a natural isomorphism

$$H^0(X, \mathcal{O}_X(mK_X)) \simeq H^0(Y, \mathcal{O}_Y(mK_Y)).$$

Proof. We will only prove this under the additional assumption such that both mK_X and mK_Y are Cartier.

Let $p: W \rightarrow X$ and $q: W \rightarrow Y$ be a common resolution of ϕ . Replacing ϕ by p we may assume that ϕ is a morphism, a log resolution of X .

Let V the indeterminacy locus of ϕ^{-1} . Suppose that ω is a pluricanonical form on X . Then $\eta = \phi_* \omega$ is a rational form on Y whose poles are concentrated on V , which is a closed subset of codimension at least two. But then η is in fact regular. Thus there is a natural map

$$H^0(X, \mathcal{O}_X(mK_X)) \rightarrow H^0(Y, \mathcal{O}_Y(mK_Y)).$$

Conversely suppose that η is a pluricanonical form on Y . By assumption,

$$K_X = \pi^* K_Y + E,$$

where $E \geq 0$ is exceptional. Then

$$\begin{aligned} \pi^* \eta &\in H^0(X, \mathcal{O}_X(m\pi^* K_Y)) \\ &\subset H^0(X, \mathcal{O}_X(m\pi^* K_Y + mE)) \\ &= H^0(X, \mathcal{O}_X(mK_X)). \end{aligned} \quad \square$$

Lemma 6.21. Let $\pi: X \rightarrow Y$ blow up a smooth variety V of codimension k , with exceptional divisor E .

Then the log discrepancy of E is equal to k .

Proof. We have

$$K_X + E = \pi^* K_Y + aE,$$

where a is the log discrepancy. Restricting to E , we have

$$K_E = (K_X + E)|_E = \pi^* K_Y|_E + aE|_E = aE|_E.$$

Let F be a general fibre. Restricting to F , we have

$$-kH = K_{\mathbb{P}^{k-1}} = K_F = aE|_F = -aH,$$

where H is the class of a hyperplane. But then $a = k$. $\square$

Lemma 6.22. *Let $(X, \Delta = \sum a_i \Delta_i)$ be a log smooth pair, where we allow some of the coefficients of Δ to be negative.*

If Δ has a component of coefficient greater than one, then set $a = -\infty$. Otherwise, let

$$a = \min_Z (k - \sum a_i),$$

where Z ranges over the irreducible components of the strata of the support of Δ , k is the codimension of Z and we sum over those components of Δ which contain Z .

Then the log discrepancy of $K_X + \Delta$ is a . In particular the log discrepancy of any pair is either at least zero, or it is $-\infty$ and if X is smooth then it is canonical.

Proof. Suppose that Δ has a component C of coefficient $1 + \epsilon$, where $\epsilon > 0$. We are going to successively blow up X along a general smooth codimension two subset of C . Thus we might as well suppose that $S = X$ is a smooth projective surface and $\Delta = (1 + \epsilon)C$, where C is a smooth curve. Suppose that we blow up $\pi: T \rightarrow S$ the point $p \in C$, with exceptional divisor E . As the log discrepancy of E with respect to K_S is 2, we have

$$K_T + E = \pi^* K_S + 2E,$$

where E is the exceptional divisor. Let D be the strict transform of C . As $\pi^* C = D + E$, it follows that

$$K_T + (1 + \epsilon)D + E = \pi^*(K_S + (1 + \epsilon)C) + (1 - \epsilon)E.$$

The log discrepancy of E is then $1 - \epsilon$. On the other hand,

$$K_T + (1 + \epsilon)D + \epsilon E = \pi^*(K_S + (1 + \epsilon)C).$$

Note that D and E are now two smooth curves, intersecting transversally at a smooth point, where D has coefficient $1 + \epsilon$ and E has coefficient ϵ . Now suppose that we blow up the intersection of D and E on T . Mutatis mutandis, a similar calculation shows that the exceptional divisor E_1 has log discrepancy $1 - 2\epsilon$ with respect to $K_T + (1 + \epsilon)D + \epsilon E$

and so also with respect to $K_S + (1 + \epsilon)C$. Moreover now we have two smooth curves intersecting transversally at a point, one with coefficient 2ϵ the other with coefficient $1 + \epsilon$. If we blow up the intersection point, then we get an exceptional divisor with log discrepancy $1 - 3\epsilon$ and so on. Continuing in this way we get exceptional divisors of log discrepancy $1 - k\epsilon$, for all $k \geq 0$. Thus the log discrepancy is $-\infty$.

Now suppose that Δ is a boundary. If we blow up Z , with exceptional divisor E , then we have

$$K_Y + E = \pi^* K_X + kE,$$

since the log discrepancy is k . Since

$$\pi_*^{-1} \Delta + (\sum a_i)E = \pi^* \Delta,$$

it follows that E has log discrepancy $k - \sum a_i$ with respect to $K_X + \Delta$.

Finally suppose that ν is some divisorial valuation. By (6.13), we may realise ν by blowing up smooth centres which intersect the support of Δ transversally. If we rewrite the equation above as

$$K_Y + \pi_*^{-1} \Delta + (\sum a_i + 1 - k)E = \pi^*(K_X + \Delta),$$

and observe that

$$\sum a_i + 1 - k \leq \min a_i,$$

since we are assuming that $a_i \leq 1$, it is easy to see that the log discrepancy is computed after one blow up. $\square$

Proof of (6.2). We will only show that the plurigenera are birational invariants; a similar argument applies to the irregularity $q(X)$. By (6.22) it follows that X and Y are canonical and we may apply (6.20). $\square$