

## 5. CHARACTERISTIC $p$

Let us consider what happens in characteristic  $p$ . The key difference between characteristic zero and characteristic  $p$  is the existence of non-trivial universal homeomorphisms:

**Definition 5.1.** *We say that a morphism  $f: X \rightarrow Y$  is a **universal homeomorphism** if for all morphisms  $g: Y' \rightarrow Y$  the morphism  $f': X' \rightarrow Y'$  induced by pullback,*

$$\begin{array}{ccc} X' & \xrightarrow{\quad} & X \\ f' \downarrow & \lrcorner & \downarrow f \\ Y' & \xrightarrow{g} & Y, \end{array}$$

*is a homeomorphism.*

**Example 5.2.** *There are two interesting examples of universal homeomorphisms, which exist in all characteristics:*

- (1) If the normalisation map  $X \rightarrow Y$  is a homeomorphism, where  $Y$  is a variety, then it is in fact a universal homeomorphism.
- (2) Let  $Y$  be a scheme of finite type over a field and let  $f: X \rightarrow Y$  be the natural inclusion, where  $X = Y_{\text{red}}$ . Then  $f$  is a universal homeomorphism.

**Definition 5.3.** *Let  $X$  be a scheme over  $\mathbb{F}_p$ . **Absolute Frobenius**  $F^X: X \rightarrow X$  is the morphism which is the identity on points, but acts as  $a \rightarrow a^q$  on functions, where  $q = p^k$  is some power of  $p$ .*

*Now suppose that  $S$  is a scheme over  $\mathbb{F}_p$  and  $f: X \rightarrow S$  is a morphism. Then there is a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{F^X} & X \\ f \downarrow & & \downarrow f \\ S & \xrightarrow{F^S} & S. \end{array}$$

*On the other hand, there is a fibre square,*

$$\begin{array}{ccc} X^{(q)} & \xrightarrow{\quad} & X \\ f^q \downarrow & \lrcorner & \downarrow f \\ S & \xrightarrow{F^S} & S. \end{array}$$

*The induced morphism  $G^X: X \rightarrow X^{(q)}$  is called **Geometric Frobenius** (over  $S$ ).*

Both  $F$  and  $G$  are universal homeomorphisms. Note that Geometric Frobenius over  $k$  is  $k$ -linear. One of the key geometric features about Frobenius is that it is everywhere not smooth, so that it kills all differentials. For example, consider

$$F: \mathbb{A}^1 \longrightarrow \mathbb{A}^1 \quad \text{where} \quad x \longrightarrow x^p$$

Then

$$F^*dx = dx^p = 0.$$

Consider the problem of trying to show that there is a divisor  $D$  on a smooth projective variety which intersects every curve positively and yet  $D$  is not ample. Mehta and Ramanathan showed that the argument in Lecture 4 works in all characteristics, provided the groundfield is uncountable. That is, they showed that there are universally stable vector bundles over appropriate (which in fact equals ordinary) smooth curves over any uncountable algebraically closed field.

But now suppose that the groundfield is the algebraic closure of a finite field  $\overline{\mathbb{F}}_p$ :

**Theorem 5.4** (Lange, Stuhle). *Suppose that  $E$  is a vector bundle of rank  $r$  over a smooth curve  $C$ , defined over  $\mathbb{F}_q$ .*

*If  $F^*E = E$  then there is a Galois étale cover  $f: C' \longrightarrow C$  such that  $f^*E$  is the trivial bundle.*

*Proof.* We just sketch the proof.

Just as in the case of characteristic zero, flat vector bundles correspond to representations of the algebraic fundamental group,

$$\rho: \pi_1^{\text{alg}}(C) \longrightarrow \text{GL}(r, \mathcal{O}_C).$$

To say that  $E$  is fixed by  $F$ , means that this representation factors as

$$\begin{array}{ccc} \pi_1^{\text{alg}}(C) & \xrightarrow{\rho} & \text{GL}(r, \mathcal{O}_C), \\ & \searrow \sigma & \nearrow i \\ & \text{GL}(r, \mathbb{F}_q) & \end{array}$$

where  $i$  is the natural inclusion. But since the group  $\text{GL}(r, \mathbb{F}_q)$  is finite, the kernel of  $\sigma$  is a normal subgroup of finite index and it follows that there is a finite cover  $f: C' \longrightarrow C$  of  $C$  for which the induced representation is trivial, which is to say that  $f^*E$  is the trivial bundle.  $\square$

Given this, it was conjectured that one could find universally stable vector bundles over the algebraic closure of any finite field. However, exactly the opposite is true:

**Theorem 5.5.** *Let  $E$  be a vector bundle of rank  $r$  over a smooth curve  $C$ .*

*Then there is a finite flat morphism  $f: C' \rightarrow C$  such that  $f^*E$  is a direct sum of line bundles.*

We will need:

**Lemma 5.6.** *Let  $X$  be a scheme of finite type over a finite field  $\mathbb{F}_q$ . Then  $|X(\mathbb{F}_q)| < \infty$ .*

*Proof.* We may assume that  $X \subset \mathbb{A}^n$  is affine. But then

$$|X(\mathbb{F}_q)| \leq |\mathbb{A}^n(\mathbb{F}_q)| = q^n. \quad \square$$

*Proof of (5.5).* By induction on the rank  $r$  of  $E$ . After passing to a finite cover, we may assume that the degree  $d$  of  $E$  is divisible by  $r$ . Tensoring by a line bundle of degree  $-d/r$ , we may then assume that  $E$  has degree zero. Consider the moduli space  $\mathcal{M}$  of semi-stable vector bundles of rank  $r$  and degree zero. This is a scheme of finite type over  $\mathbb{F}_q$ . Thus

$$|\mathcal{M}(\mathbb{F}_q)| < \infty.$$

Note that this implies that there are only finitely many semi-stable vector bundles. Indeed if  $[E_1] = [E_2] \in \mathcal{M}(\mathbb{F}_q)$  for two semi-stable vector bundles  $E_1$  and  $E_2$ , then  $E_1$  and  $E_2$  both have the same filtration by stable bundles. But since extensions

$$0 \rightarrow V \rightarrow E \rightarrow Q \rightarrow 0,$$

are classified by points of

$$H^1(C, V \otimes Q^*),$$

which is a finite dimensional vector space over  $\mathbb{F}_q$ , any point in  $\mathcal{M}(\mathbb{F}_q)$  corresponds to finitely many semi-stable vector bundles.

A moment's thought will convince the reader that there are then two cases:

- (1)  $F^{(k+l)*}E = F^{k*}E$ , for some  $k$  and  $l > 0$ , or
- (2)  $E' = F^{k*}E$  is not stable, for some  $k > 0$ .

Suppose (1) holds. Let  $E' = F^{k*}E$ . Then  $E'$  is stable under some power  $F^l$  of Frobenius. But then by (5.4) there is a finite étale cover such that  $E'$  pulls back to the trivial bundle.

Thus we may assume that (2) holds. Replacing  $E$  by  $E'$  we may assume that  $E$  is not stable. Let  $E \rightarrow Q$  be a destabilising quotient, with kernel  $V$ . By induction on the rank, after passing to a finite morphism, we may assume that  $V$  and  $Q$  are direct sums of line bundles  $L_1, L_2, \dots, L_s$  and  $M_1, M_2, \dots, M_t$ . Replacing  $Q$  be a quotient, we may

then assume that the degree of each  $L_i$  is strictly less than the degree of each  $M_j$ . Now the extension class of the short exact sequence

$$0 \longrightarrow V \longrightarrow E \longrightarrow Q \longrightarrow 0,$$

lies in

$$H^1(C, V \otimes Q^*) = \bigoplus H^1(C, L_i^* \otimes M_j) = \bigoplus H^0(C, \omega_C \otimes L_i \otimes M_j^*)$$

Consider what happens when we apply Frobenius. This has the effect of replacing  $L_i$  by  $L_i^{\otimes q}$  and  $M_j$  by  $M_j^{\otimes q}$ . Thus we may assume that

$$\deg L_i \otimes M_j^* < -(2g - 2).$$

But then

$$H^1(C, V \otimes Q^*) = 0,$$

so that the sequence splits and  $E$  is a direct sum of line bundles.  $\square$

**Theorem 5.7** (Totaro). *There is a smooth rational surface  $S$  and a smooth curve  $C$  of genus 2 inside  $S$  such that  $C^2 = 0$  and yet  $C$  is not semiample, over any field.*

**Definition 5.8.** *Let  $X$  be a projective variety and let  $D$  be a  $\mathbb{Q}$ -Cartier divisor on  $X$ .*

*The **stable base locus** of  $D$  is the intersection of the base locus of all integral multiples of  $D$ ,*

$$\text{SBs}(D) = \bigcap_{m \in \mathbb{N}} \text{Bs}(mD),$$

*where  $k$  is the smallest natural number such that  $kD$  is integral.*

Zariski proved the following beautiful result, which is valid in all characteristics:

**Theorem 5.9** (Zariski). *Let  $X$  be a normal projective variety and let  $D$  be a  $\mathbb{Q}$ -Cartier divisor on  $X$ .*

*If the stable base locus of  $|D|$  is a finite set of points then  $D$  is semiample.*

Note that if the stable base locus is a finite set of points, then  $D$  is nef. Indeed if  $\Sigma$  is any curve, then pick a point  $x \in \Sigma$  not in the stable base locus of  $D$ . Then we may find  $D \sim B \geq 0$  not containing  $x$  and

$$D \cdot \Sigma = B \cdot \Sigma \geq 0.$$

On the other hand, by a similar reasoning we may find  $D_1, D_2, \dots, D_n$  whose intersection is a finite set of points (or empty) so that either

$$D^n > 0,$$

or the stable base locus is empty. So to prove Zariski's theorem, we may assume that  $D$  is big and nef.

We also have the following result due to Fujita, which is again valid in all characteristics:

**Theorem 5.10** (Fujita). *Let  $X$  be a normal projective variety and let  $D$  be a  $\mathbb{Q}$ -Cartier divisor on  $X$ , with stable base locus  $Z$ .*

*If  $D|_Z$  is ample then  $D$  is semiample.*

Note that this is a stronger version of (5.9). Note also that if  $D|_Z$  is ample then  $D$  is clearly nef, and if  $Z$  is non-empty, then  $D$  is big.

By contrast we have a much stronger result in characteristic  $p$ :

**Theorem 5.11** (Keel). *Let  $X$  be a projective scheme over a field of characteristic  $p$  and let  $D$  be a big and nef Cartier divisor on  $X$ . Let  $Z \supset \mathbb{E}(D)$  be any closed subset which contains the exceptional locus.*

*If  $\sigma \in H^0(Z, \mathcal{O}_Z(D))$  then  $\sigma^q$  lifts to  $\tau \in H^0(X, \mathcal{O}_X(qD))$ , for some power  $q = p^k$  of  $p$ .*

*In particular  $D$  is semiample if and only if  $D|_{\mathbb{E}(D)}$  is semiample.*

**Definition 5.12.** *Let  $X$  be a projective scheme and let  $D$  be a  $\mathbb{Q}$ -Cartier divisor. The **exceptional locus** of  $D$  is the union of all subvarieties  $Z$  for which  $D|_Z$  is not big,*

$$\mathbb{E}(D) = \bigcup \{ Z \subset X \mid (D^k \cdot Z) = 0. \}.$$

**Definition 5.13.** *Let  $X$  be a projective scheme and let  $D$  be a  $\mathbb{Q}$ -Cartier divisor.*

*The **augmented stable base locus** of  $D$  is the stable base locus of  $D - \epsilon A$ , where  $A$  is ample and  $\epsilon > 0$  is sufficiently small,*

$$B_+(D) = \bigcap_{\epsilon > 0} \text{SBs}(D - \epsilon A).$$

Note that if

$$\eta < \epsilon \quad \text{then} \quad \text{SBs}(D - \eta A) \subset \text{SBs}(D - \epsilon A),$$

so that

$$\bigcap_{\epsilon > 0} \text{SBs}(D - \epsilon A) = \text{SBs}(D - \eta A)$$

for any  $\eta > 0$  sufficiently small.

In fact the exceptional locus and the augmented stable base locus coincide for nef divisors:

**Lemma 5.14.** *Let  $X$  be a projective scheme and let  $D$  be a  $\mathbb{Q}$ -Cartier divisor.*

(1)

$$\mathbb{E}(D) \subset B_+(D).$$

(2)  $B_+(D)$  and  $\mathbb{E}(D)$  are closed subsets.

(3) If  $D$  is nef and either the characteristic is positive or  $X$  is smooth then

$$B_+(D) = \mathbb{E}(D).$$

In particular  $\mathbb{E}(D)$  is a closed subset of the augmented stable base locus of  $D$ .

*Proof.* Suppose that  $Z \not\subset B_+(X)$  is a closed irreducible subset. Let  $A$  be an ample divisor. Then we may find  $0 \leq C \sim_{\mathbb{Q}} D - \epsilon A$ , for some  $\epsilon > 0$ , such that  $Z$  is not contained in the support of  $C$ . But then

$$D|_Z = \epsilon A|_Z + C|_Z = A' + B',$$

where  $A'$  is ample and  $B' \geq 0$ . But then  $D|_Z$  is big so that  $Z \not\subset \mathbb{E}(D)$ . Hence (1).

Note that  $B_+(D)$  is certainly closed, since it is the intersection of closed subsets. We may write  $X = X_1 \cup X_2$  so that  $D$  restricted to every component of  $X_1$  is big and its restriction to every component of  $X_2$  is not big. Clearly

$$X_2 \subset \mathbb{E}(D).$$

On the other hand, by Kodaira's Lemma, we may write

$$D|_{X_1} = D_1 \sim_{\mathbb{Q}} A + B,$$

where  $A$  is ample and  $B \geq 0$ . Let  $Z \subset X_1$  be a closed subvariety not contained in  $B$ . Then

$$D|_Z \sim_{\mathbb{Q}} A|_Z + B|_Z = A' + B',$$

where  $A'$  is ample and  $B' \geq 0$ . In particular  $D|_Z$  is big and so  $Z \not\subset \mathbb{E}(D)$ . But then

$$\mathbb{E}(D) = \mathbb{E}(D_1) \cup X_2.$$

Hence (2).

(3) is a theorem of Nakamaye in characteristic zero and follows from (5.11) in characteristic  $p$ .  $\square$

Note that the augmented stable base locus of  $D$  is empty if and only if  $D$  is ample. In particular Fujita's result says that  $D$  is semiample if  $D|_Z$  has empty augmented stable base locus, where  $Z$  is the base locus.

One of the key results we will need is interesting in its own right:

**Proposition 5.15.** *Let  $f: X \rightarrow Y$  be a finite, universal homeomorphism between schemes of finite type over a field of characteristic  $p$ , and let  $D$  be a Cartier divisor on  $Y$ .*

*If  $\sigma \in H^0(X, \mathcal{O}_X(f^*D))$  then for some power  $q = p^k$  of  $p$  we may lift  $\sigma^q$  to  $\tau \in H^0(Y, \mathcal{O}_Y(qD))$ .*

*In particular  $D$  is semiample if and only if  $f^*D$  is semiample.*

**Example 5.16.** *Let  $C \subset \mathbb{P}^2$  be an irreducible singular cubic.*

Note that the normalisation of  $C$  is isomorphic to  $\mathbb{P}^1$ ,

$$f: \mathbb{P}^1 \rightarrow C.$$

Consider  $\text{Pic}^0(C)$ . If we take any line bundle  $L \in \text{Pic}^0(C)$  then  $f^*L$  is a line bundle of degree zero. Thus  $f^*L$  is the trivial line bundle.

There are two cases. If  $C$  is a cuspidal cubic then  $f$  is a universal homeomorphism. In this case  $\text{Pic}^0(C) \simeq \mathbb{G}_a$ , the additive group of the underlying field. In characteristic zero every non-zero element of  $\mathbb{G}_a$  has infinite order. The only torsion line bundles are trivial. In characteristic  $p$  every element has order  $p$ , as predicted by (5.15).

By contrast, if  $C$  is a nodal cubic then  $\text{Pic}^0(C) \simeq \mathbb{G}_m$ , the multiplicative group of the underlying field. There are lots of torsion line bundles (corresponding to roots of unity in the underlying field) but if the underlying field is anything but the algebraic closure of a finite field then there are also line bundles of degree zero which are not torsion. Of course  $f$  is not an isomorphism in this case.

(5.15) is a simple consequence of the following result:

**Definition-Lemma 5.17.** *Let  $f: X \rightarrow Y$  be a finite, universal homeomorphism between schemes of finite type over a field of characteristic  $p$ .*

*Then there is a universal homeomorphism  $\bar{f}: Y \rightarrow X^q$  such that  $G^X = \bar{f} \circ f$ , for any  $q$  sufficiently large. This factorisation is functorial, if  $q$  is sufficiently large, in the following sense. Suppose that we are given a commutative square:*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ g \downarrow & & \downarrow h \\ X & \xrightarrow[f]{7} & Y. \end{array}$$

Then there is a commutative square

$$\begin{array}{ccc} Y' & \xrightarrow{\bar{f}'} & X'^q \\ \downarrow h & & \downarrow g^q \\ Y & \xrightarrow{\bar{f}} & X^q. \end{array}$$

*Proof.* By functoriality it suffices to prove this result when  $X = \text{Spec } B$  and  $Y = \text{Spec } A$  are affine schemes. Thus we have a  $k$ -algebra homomorphism  $\phi: A \rightarrow B$ . We can factor  $\phi$  into a surjection  $A \rightarrow B_1$  followed by an inclusion  $B_1 \rightarrow B$ .

So, we first consider the inclusion  $B_1 \rightarrow B$ . We want to show that there is a  $q$  such that  $B^q \subset B_1$ , where  $B^q$  is the algebra generated by the  $q$ th powers. We proceed by Noetherian induction. If  $B_1$  is Artinian, the result is clear. Thus

$$B_1 \subset B_1 B^q,$$

is an isomorphism at every generic point, for  $q$  sufficiently large. Let  $I \triangleleft B_1$  be the conductor ideal of this extension,

$$I = \{ b \in B_1 \mid b(B_1 B^q) \subset B_1 \},$$

so that the locus where  $I$  vanishes is where the map on rings is not an isomorphism.

Then by induction

$$((B_1 B^q)/I)^{q'} \subset B_1/I.$$

for all  $q'$  sufficiently large. But then we have

$$B^{(qq')} \rightarrow B^{qq'} \subset (B_1 B^q)^{q'} \subset B_1.$$

Now suppose that  $A \rightarrow B_1$  is a surjection. Then the kernel  $I \triangleleft A$  is nilpotent  $I^k = 0$ , some  $k$ . Pick  $q > k$ . Given  $b \in B_1$ , let  $a$  be any element of the preimage. Then  $a^q$  does not depend on what element we choose; thus we get a canonical factorisation

$$B_1^{(q)} \rightarrow A \rightarrow B_1$$

Combining the two maps, gives us the result.  $\square$

**Example 5.18.** Consider the standard cusp

$$C = (y^2 - x^3 = 0) \subset \mathbb{A}_k^2$$

The normalisation map

$$f: \mathbb{A}^1 \rightarrow C$$

is a finite universal homomorphism. The normalisation map is given by

$$t \rightarrow (t^2, t^3) = (x, y).$$

This corresponds to the inclusion of rings

$$k[t^2, t^3] \subset k[t].$$

Note that

$$k[t^p] \subset k[t^2, t^3] \subset k[t]$$

and this says that the normalisation map factors through

$$\mathbb{A}^1 \longrightarrow \mathbb{A}^1 \quad \text{given by} \quad t \longrightarrow t^p.$$

*Proof of (5.15).* By (5.17) we may find  $\bar{f}: Y \longrightarrow X$ , so that  $\bar{f} \circ f = F^X$ , for some  $q = p^k$ , a power of  $p$ . This gives rise to a commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{O}_X(f^*D)) & \xrightarrow{\bar{f}^*} & H^0(Y, \mathcal{O}_Y(qD)) \\ & \searrow F^* & \downarrow f^* \\ & & H^0(X, \mathcal{O}_X(qf^*D)). \end{array}$$

Let  $\tau = \bar{f}^*\sigma \in H^0(Y, \mathcal{O}_Y(qD))$ . Then  $f^*(\tau) = \sigma^q$ .  $\square$

*Proof of (5.11).* By (5.6) applied to the inclusion  $X_{\text{red}} \longrightarrow X$  we may assume that  $X$  is a variety. We may write  $X = X_1 \cup X_2$ , where the restriction of  $D$  to every component of  $X_1$  is big and the restriction of  $D$  to every component of  $X_2$  is not big. There is an exact sequence

$$0 \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_{X_1}(D) \oplus \mathcal{O}_{X_2}(D) \longrightarrow \mathcal{O}_{X_1 \cap X_2}(D) \longrightarrow 0.$$

Taking cohomology, it follows that sections  $\tau$  of  $H^0(X, \mathcal{O}_X(qD))$  correspond to pairs of sections  $\tau_1$  of  $H^0(X_1, \mathcal{O}_{X_1}(qD))$  and  $\tau_2$  of  $H^0(X_2, \mathcal{O}_{X_2}(qD))$ , which agree on the intersection,  $(\tau_1 - \tau_2)|_{X_1 \cap X_2} = 0$ . Since  $X_2 \subset Z$ , it suffices to show that we can lift  $\sigma^q$  from  $X_1 \cap Z$  to  $\tau_1$  on  $X_1$ , for some power  $q = p^k$  of  $p$ , since  $\tau_2 = \sigma^q|_{X_2}$  will automatically agree with  $\tau_1$  on  $X_1 \cap X_2$ .

Thus we may assume that  $D$  is big. By Kodaira's Lemma  $D \sim_{\mathbb{Q}} A + E$ , where  $A$  is ample and  $E \geq 0$ . Then  $Z \subset E$ . By Noetherian induction, we may assume that  $Z$  coincides with the support of  $E$ . Replacing  $D$  by a multiple we may assume that  $Z = E$ . Pick a positive integer  $r$  such that  $rD$ ,  $rA$  and  $rE$  are Cartier,  $rD \sim rA + rE$ . Replacing  $D$  by  $rD$  and  $A$  by  $rA$  we may assume that  $D = A + rE$ . Note that as

$$\begin{aligned} r(A + (r-1)E) &= rA + r(r-1)E \\ &= (r-1)A + r(r-1)E + A \\ &= (r-1)(A + rE) + A \\ &= (r-1)D + A, \end{aligned}$$

it follows that  $A + (r - 1)E$  is ample.

Given  $q$ , let  $E_q$  be the  $q$ th thickening of  $E$ . Note that by the proof of (5.15), we may lift  $\sigma^q$  to  $\sigma_q \in \mathcal{O}_{E_q}(qD)$ . There is a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(qD - qE) \longrightarrow \mathcal{O}_X(qD) \longrightarrow \mathcal{O}_{E_q}(qD) \longrightarrow 0.$$

Now

$$\begin{aligned} qD - qE &= q(A + rE) - qE \\ &= q(A + (r - 1)E). \end{aligned}$$

It follows that

$$H^1(X, \mathcal{O}_X(qD - qE)) = 0,$$

for  $q$  large enough, by Serre vanishing. Thus there are no obstructions to lifting  $\sigma'$  to  $\tau \in H^0(X, \mathcal{O}_X(qD))$ .  $\square$

**Example 5.19.** *Pick an elliptic curve defined over  $\mathbb{Q}$  whose Mordell-Weil group has rank at least ten.*

Pick ten rational points  $p_1, p_2, \dots, p_{10} \in E \subset \mathbb{P}^2$ , whose images are independent from the class of a hyperplane in the Mordell-Weil group. Let  $\pi: S \longrightarrow \mathbb{P}^2$  be the blow up of these ten points. Let  $D = 10\pi^*H - 3\sum E_i$ , where  $E_1, E_2, \dots, E_{10}$  are the ten exceptional divisors. Then  $D$  is a big and nef divisor, whose restriction to  $\Sigma$ , the strict transform of  $C$ , has degree zero, but is not torsion. Let  $f: S \longrightarrow T$  contract  $\Sigma$ . Then we may realise this whole picture over  $\text{Spec } \mathbb{Z}$ .

Over a finite field,  $D|_\Sigma$  is torsion. But then by (5.11) the divisor  $D$  is semiample, when we reduce modulo every prime  $p$  and yet it is not semiample. Indeed  $T$  is projective modulo every prime and yet it is not projective.

To end this section, let us first observe that both of these characteristic  $p$  results rely on the fact that we can evaporate cohomology:

**Definition 5.20.** *Let  $X$  be a scheme and let  $\mathcal{F}$  be a sheaf. We say that  $f: X \longrightarrow Y$  **evaporates**  $\alpha \in H^*(X, \mathcal{F})$  if  $f^*\alpha = 0$ .*

**Theorem 5.21** (Hochester, Huneke). *Let  $X$  be a projective scheme over a field of characteristic  $p$ . Let  $D$  be an integral and ample divisor on  $X$ .*

*Then there is a finite surjective morphism of degree a power  $q = p^k$  of  $p$  which evaporates every element of the bigraded ring*

$$T(X, D) = \bigoplus_{i > 0, m \geq 0} H^i(X, \mathcal{O}_X(mD)).$$

*Proof.* By Serre vanishing, the vector space  $T(X, D)$  is finite dimensional. It therefore suffices to prove this result element by element. If

$m > 0$  just take  $f = F^X$ , where  $q$  is sufficiently large and apply Serre vanishing (the  $q$ -linear map  $F^*$  is zero, simply as the image vector space is zero).

So we may assume that  $m = 0$ . I only prove the case  $i = 1$ . Consider the action of  $F^X$  on  $H^1(X, \mathcal{O}_X)$ . This action is  $q$ -linear, meaning that it is additive and

$$F^*(\lambda\alpha) = \lambda^q F^*(\alpha).$$

Over an algebraically closed field, any  $q$ -linear endomorphism of a finite dimensional vector space  $V$ , decomposes the vector space into a direct sum:

$$V = V_n + V_s,$$

where the action of  $F$  is nilpotent on  $V_n$  (that is some power is zero) and  $V_s$  has a basis of vectors  $\alpha_1, \alpha_2, \dots, \alpha_m$  such that  $F^*\alpha = \alpha$ . By a result of Serre, these elements correspond to cyclic covers of degree  $p$ ,  $f_i: Y_i \rightarrow X$ , and almost by definition  $f_i^*\alpha_i = 0$ .  $\square$

If  $X$  is an abelian variety and  $V_s = H^1(X, \mathcal{O}_X)$  then we say that  $X$  is **ordinary**. At the opposite extreme, there are examples where  $V_n = H^1(X, \mathcal{O}_X)$ .

**Question 5.22.** *Let  $X$  be a projective variety over  $\mathbb{Z}$ . If  $E$  is a vector bundle over  $X$ , then is the set*

$$\{p \in \mathbb{Z} \mid \text{there is a smooth finite cover } f: Y_p \rightarrow X_p \text{ such that } f_p^*E_p \text{ is a direct sum of line bundles}\},$$

*infinite (dense?).*

If  $\alpha \in H^1(X, \mathcal{O}_X)$ , then we get an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow \mathcal{O}_X \rightarrow 0.$$

Then  $f: Y_p \rightarrow X_p$  splits  $E_p$  if and only if  $\alpha \in V_s$ . If the set above were dense (and not just infinite), then (5.22) would imply that the set of primes for which  $X$  is ordinary is dense (which is a conjecture).