

3. AMPLE AND SEMIAMPLE

We recall some very classical algebraic geometry. Let D be an integral Weil divisor. Provided $h^0(X, \mathcal{O}_X(D)) > 0$, D defines a rational map:

$$\phi = \phi_D: X \dashrightarrow Y.$$

The simplest way to define this map is as follows. Pick a basis $\sigma_1, \sigma_2, \dots, \sigma_m$ of the vector space $H^0(X, \mathcal{O}_X(D))$. Define a map

$$\phi: X \longrightarrow \mathbb{P}^{m-1} \quad \text{by the rule} \quad x \longrightarrow [\sigma_1(x) : \sigma_2(x) : \dots : \sigma_m(x)].$$

Note that to make sense of this notation one has to be a little careful. Really the sections don't take values in $\mathbb{C}$, they take values in the fibre L_x of the line bundle L associated to $\mathcal{O}_X(D)$, which is a 1-dimensional vector space (let us assume for simplicity that D is Cartier so that $\mathcal{O}_X(D)$ is locally free). One can however make local sense of this morphism by taking a local trivialisation of the line bundle $L|_U \simeq U \times \mathbb{C}$. Now on a different trivialisation one would get different values. But the two trivialisations differ by a scalar multiple and hence give the same point in $\mathbb{P}^{m-1}$. In fact if one wants to be very cheap, since we are just trying to define a rational map, it suffices to fix one open set and define this map on this open set.

However a much better way to proceed is as follows.

$$\mathbb{P}^{m-1} \simeq \mathbb{P}(H^0(X, \mathcal{O}_X(D))^*).$$

Given a point $x \in X$, let

$$H_x = \{ \sigma \in H^0(X, \mathcal{O}_X(D)) \mid \sigma(x) = 0 \}.$$

Then H_x is a hyperplane in $H^0(X, \mathcal{O}_X(D))$, whence a point of

$$\phi(x) = [H_x] \in \mathbb{P}(H^0(X, \mathcal{O}_X(D))^*).$$

Note that ϕ is not defined everywhere. The problem in either description is that the sections of $H^0(X, \mathcal{O}_X(D))$ might all vanish at x .

Definition 3.1. *Let X be a normal variety and let D be an integral Weil divisor. The **complete linear system***

$$|D| = \{ D' \mid D' \geq 0, D' \sim D \}.$$

*The **base locus** of $|D|$ is the intersection of all of the elements of $|D|$. We say that $|D|$ is **base point free** if the base locus is empty.*

In fact, given a section $\sigma \in H^0(X, \mathcal{O}_X(D))$ the zero locus D' of σ defines an element of $|D|$ and every element of $|D|$ arises in this fashion.

Now assume that the base locus has codimension two or more, that is, assume that there no divisor is in the base locus. In this case

the base locus of $|D|$ is precisely the locus where ϕ_D is not defined. Thus $|D|$ is base point free if and only if ϕ_D is a morphism. In this case the linear system can be recovered by pulling back the hyperplane sections of $Y \subset \mathbb{P}^{m-1}$ and in fact $\mathcal{O}_X(D) = \phi^* \mathcal{O}_{\mathbb{P}^n}(1)$.

Definition 3.2. We say that a $\mathbb{Q}$ -divisor D on a normal variety is **semiample** if $|mD|$ is base point free for some $m \in \mathbb{N}$. We say that an integral divisor D is **very ample** if $\phi: X \rightarrow \mathbb{P}^n$ defines an embedding of X . We say that D is **ample** if mD is very ample for some $m \in \mathbb{N}$.

Theorem 3.3 (Serre-Vanishing). Let X be a normal projective variety and let D be a Cartier divisor on X .

The following are equivalent:

- (1) D is ample.
- (2) For every coherent sheaf $\mathcal{F}$ on X , there is a positive integer m such that

$$H^i(X, \mathcal{F}(mD)) = 0,$$

for all $m \geq m_0$ and $i > 0$ (and these cohomology groups are finite dimensional vector spaces).

- (3) For every coherent sheaf $\mathcal{F}$ on X , there is a positive integer m_0 such that the natural map

$$H^0(X, \mathcal{F}(mD)) \otimes \mathcal{O}_X \rightarrow \mathcal{F}(mD),$$

is surjective, for all m divisible by m_0 .

Proof. Suppose that D is ample. Then $\phi_{kD} = i: X \rightarrow \mathbb{P}^n$ defines an embedding of X into projective space, for some $k \in \mathbb{N}$. Let $\mathcal{F}$ be a coherent sheaf on X . Let $\mathcal{F}_j = \mathcal{F}(jD)$, $0 \leq j \leq k-1$. Let $m \in \mathbb{N}$. Then we may write $m = m'k + j$, for some $0 \leq j \leq k-1$. In this case

$$H^i(X, \mathcal{F}(mD)) = H^i(X, \mathcal{F}_j(m'kD)).$$

Thus replacing D by kD we may assume that $D = i^*H$ is very ample. In this case

$$H^i(X, \mathcal{F}(mD)) = H^i(\mathbb{P}^n, i_*\mathcal{F}(mH)).$$

Thus replacing X by $\mathbb{P}^n$, $\mathcal{F}$ by $i_*\mathcal{F}$ and D by H we may assume that $X = \mathbb{P}^n$.

Now every sheaf on $\mathbb{P}^n$ has a finite length resolution, where each term is a direct sum of line bundles. We may break this resolution into a sequence of short exact sequences,

$$0 \rightarrow \mathcal{F}_{i+1} \rightarrow \mathcal{E}_i \rightarrow \mathcal{F}_i \rightarrow 0,$$

where $0 \leq i < k$, $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{E}_i$ is a direct sum of line bundles. Since we have shown that $A_{n-1}(\mathbb{P}^n) = \mathbb{Z}$ it follows that each line bundle on $\mathbb{P}^n$ is of the form $\mathcal{O}_{\mathbb{P}^n}(aH)$ for some $a \in \mathbb{Z}$ (it is customary to drop the

H). Twisting by a large multiple of H we may assume that each $a > 0$ is large. By direct computation

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(a)) = 0,$$

for $i > 0$. Taking the long exact sequence associated to the short exact sequence gives

$$H^j(\mathbb{P}^n, \mathcal{F}_i) \simeq H^{j+1}(\mathbb{P}^n, \mathcal{F}_{i+1}),$$

and so (1) implies (2) by induction on i .

Suppose that (2) holds. Let $p \in X$ be a point of X and let $\mathbb{C}_p$ be the skyscraper sheaf supported at p , with stalk $\mathbb{C}$. Then there is an exact sequence

$$0 \longrightarrow \mathcal{I}(mD) \longrightarrow \mathcal{F}(mD) \longrightarrow \mathcal{F}(mD) \otimes \mathbb{C}_p \longrightarrow 0,$$

where $\mathcal{I}(mD)$ is a coherent sheaf. Since by assumption

$$H^1(X, \mathcal{I}(mD)) = 0,$$

for m sufficiently large, taking the long exact sequence of cohomology, it follows that

$$H^0(X, \mathcal{F}(mD)) \longrightarrow H^0(X, \mathcal{F}(mD) \otimes \mathbb{C}_p),$$

is surjective for m sufficiently large. But then

$$H^0(X, \mathcal{F}(mD)) \otimes \mathcal{O}_X \longrightarrow \mathcal{F}(mD),$$

is surjective in a neighbourhood of p by Nakayama's Lemma. As X is compact, it follows that (2) implies (3).

Suppose that (3) holds. Pick m_0 such that

$$H^0(X, \mathcal{O}_X(mD)) \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_X(mD),$$

is surjective for all $m \geq m_0$. Since there is a surjection

$$\mathcal{O}_X(mD) \longrightarrow \mathbb{C}_p(mD),$$

it follows that there is a surjection,

$$H^0(X, \mathcal{O}_X(mD)) \otimes \mathcal{O}_X \longrightarrow \mathbb{C}_p(mD).$$

But then we may find a section $\sigma \in H^0(X, \mathcal{O}_X(mD))$ not vanishing at x . This gives an element $D' \in |mD|$ not containing x . It follows that mD is base point free for all m sufficiently large and divisible. Thus D is semiample.

Let $x \neq y$ be two points of X and let $\mathcal{I}_x$ be the ideal sheaf of x . Pick a positive integer m_0 such that

$$H^0(X, \mathcal{I}_x(mD)) \otimes \mathcal{O}_X \longrightarrow \mathcal{I}_x(mD),$$

is surjective for all $m \geq m_0$. As above, it follows that

$$H^0(X, \mathcal{I}_x(mD)) \otimes \mathcal{O}_X \longrightarrow \mathcal{I}_x \otimes \mathbb{C}_y(mD) = \mathbb{C}_y(mD),$$

is surjective. But then we may an element of $|mD|$ passing through x but not through y . Thus $\phi_{mD}(x) \neq \phi_{mD}(y)$.

Let S_m be the set of points x such that there is some $y \neq x$ such that if $D' \in |mD|$ and D' contains x then D' does not contain y . Note that if $x \notin S_m$ then $x \notin S_{mk}$, for all $k \in \mathbb{N}$. Indeed if $x \notin S_m$ then for every $y \neq x$ there is a divisor $D' \in |mD|$ containing x not containing y . But then $kD' \in |kmD|$ is a divisor containing x but not containing y . Thus $x \notin S_{mk}$. By Noetherian induction B_m is empty for m sufficiently large and divisible and so ϕ_{mD} separates points.

Finally pick a length two scheme $z \subset X$ supported at x . This determines a surjection,

$$\mathcal{I}_z(mD) \longrightarrow \mathcal{O}_z(mD)$$

Composing, we get a surjection

$$H^0(X, \mathcal{I}_z(mD)) \otimes \mathcal{O}_X \longrightarrow \mathbb{C}_z(mD),$$

By a repeat of the Noetherian argument given above, if m is sufficiently large and divisible then this map is surjective for all irreducible length two schemes. On the other hand, the Zariski tangent space of any scheme is the union of the irreducible length two subschemes of X , and so the map ϕ_{mD} is an injective immersion (that is, it is injective on tangent spaces). But then ϕ_{mD} is an embedding (it is a straightforward application of Nakayama's Lemma to check that the implicit function argument extends to the case of singular varieties). Thus (3) implies (1). $\square$

More generally if X is a scheme, we will use (2) of (3.3) as the definition of ample:

Lemma 3.4. *Let $f: X \longrightarrow Y$ be a morphism of projective schemes. Let D be a $\mathbb{Q}$ -Cartier divisor on Y .*

- (1) *If D is ample and f is finite then f^*D is ample.*
- (2) *If f is surjective and f^*D is ample (this can only happen if f is finite) then D is ample.*

Proof. We may suppose that D is Cartier.

Suppose that D is ample and let $\mathcal{F}$ be a coherent sheaf on X . As f is finite, we have

$$H^i(X, \mathcal{F}(mf^*D)) = H^i(Y, f_*\mathcal{F}(mD)),$$

which is zero for m sufficiently large. Thus f^*D is ample. This proves (1).

Now suppose that f is surjective and f^*D is ample. We first prove (2) in the special case when $X = Y_{\text{red}}$ and f is the natural inclusion.

Suppose that f^*D is ample. Let $\mathcal{I}_X$ be the ideal sheaf of X in Y . Then $\mathcal{I}_X^k = 0$ for some $k > 0$, which we may assume to be minimal. We proceed by induction on k . If $k = 1$ then f is an isomorphism and there is nothing to prove. Otherwise $k > 1$. Let $\mathcal{F}$ be a coherent sheaf on Y . Then there is an exact sequence

$$0 \longrightarrow \mathcal{H}(mD) \longrightarrow \mathcal{F}(mD) \longrightarrow \mathcal{G}(mD) \longrightarrow 0,$$

where $\mathcal{G} = \mathcal{F} \otimes \mathcal{O}_X$ and $\mathcal{H}(mD)$ is coherent. Then for m sufficient large,

$$h^i(X, \mathcal{G}(mD)) = 0,$$

and so take the long exact sequence of cohomology we get

$$h^i(Y, \mathcal{F}(mD)) = h^i(Y, \mathcal{H}(mD)),$$

On the other hand, $\mathcal{H}$ is a coherent sheaf supported on the proper subscheme $X' \subset X$ defined by $\mathcal{I}^{k-1}$, and we are done by induction on k .

Let $X' = X_{\text{red}}$ and $Y' = Y_{\text{red}}$. Since taking the reduction is a functor, there is a commutative square

$$\begin{array}{ccc} X' & \longrightarrow & X \\ f' \downarrow & & \downarrow f \\ Y' & \longrightarrow & Y, \end{array}$$

where $f' = f_{\text{red}}$. We may pullback D to X' either way around the square and we get the same answer both ways. It follows that to prove that D is ample, it suffices to prove that $D' = D_{\text{red}}$ is ample. Thus, replacing $f: X \rightarrow Y$ by $f': X' \rightarrow Y'$, we may assume that X and Y are projective varieties.

Now suppose that $Y = Y_1 \cup Y_2$ is the union of two closed subvarieties, let X be their disjoint union and let $f: X \rightarrow Y$ be the obvious morphism. If $\mathcal{F}$ is any coherent sheaf, then there is an exact sequence

$$0 \longrightarrow \mathcal{F}(mD) \longrightarrow f^*\mathcal{F}(mD) = \mathcal{F}_1(mD) \oplus \mathcal{F}_2(mD) \longrightarrow \mathcal{G}(mD) \longrightarrow 0,$$

where $\mathcal{F}_i = \mathcal{F} \otimes \mathcal{O}_{Y_i} = \mathcal{F}|_{Y_i}$ is the restriction of $\mathcal{F}$ to Y_i and $\mathcal{G}$ is a sheaf supported on $Z = Y_1 \cap Y_2$. By (1) applied to either of the inclusions $Z \rightarrow Y_i$, $D|_Z$ is ample. Moreover for m sufficiently large both of the natural maps,

$$H^0(Y_i, \mathcal{F}_i(mD)) \longrightarrow H^0(Z, \mathcal{G}(mD)),$$

are surjective. Taking the long exact sequence of cohomology, we see that

$$H^i(X, \mathcal{F}(mD)) = 0,$$

is zero, for $i > 0$ and m sufficiently large, since

$$H^0(Y_1, \mathcal{F}_1(mD)) \oplus H^0(Y_2, \mathcal{F}_2(mD)) \longrightarrow H^0(Z, \mathcal{G}(mD)),$$

is surjective, for m sufficiently large.

Using the same argument and the same square as before (but where X' and Y' are now the irreducible components of X and Y) we may assume that X and Y are integral. Now the proof proceeds by induction on the dimension of X (and so of Y).

Let $\mathcal{F}$ be a sheaf on Y .

Claim 3.5. *There is a sheaf $\mathcal{G}$ on X and a map*

$$f_*\mathcal{G} \longrightarrow \bigoplus_{i=1}^k \mathcal{F},$$

which is an isomorphism over an open subset of Y , where k is the degree of the map f .

Proof of (3.5). Note that k is the degree of the field extension $K(X)/K(Y)$. Pick an affine subset U of X and pick sections $s_1, s_2, \dots, s_k \in \mathcal{O}_U$ which generate the field extension $K(X)/K(Y)$. Let $\mathcal{M}$ be the coherent sheaf generated by the sections $s_1, s_2, \dots, s_k \in K(Y)$. Then there is a morphism of sheaves

$$\bigoplus_{i=1}^k \mathcal{O}_Y \longrightarrow f_*\mathcal{M},$$

which is an isomorphism over an open subset of Y . Taking $\mathrm{Hom}_{\mathcal{O}_Y}$, we get a morphism of sheaves

$$\mathrm{Hom}_{\mathcal{O}_Y}(f_*\mathcal{M}, \mathcal{F}) \longrightarrow \mathrm{Hom}_{\mathcal{O}_Y}\left(\bigoplus_{i=1}^k \mathcal{O}_Y, \mathcal{F}\right) \simeq \bigoplus_{i=1}^k \mathcal{F}.$$

Finally observe that

$$\mathrm{Hom}(f_*\mathcal{M}, \mathcal{O}_Y) = f_*\mathcal{G},$$

for some coherent sheaf $\mathcal{G}$ on X . □

Thus there is an exact sequence

$$0 \longrightarrow \mathcal{K}(mD) \longrightarrow f_*\mathcal{G}(mD) \longrightarrow \bigoplus_{i=1}^k \mathcal{F}(mD) \longrightarrow \mathcal{Q}(mD) \longrightarrow 0,$$

where $\mathcal{K}$ and $\mathcal{Q}$ are defined to preserve exactness. We may break this exact sequence into two parts,

$$0 \longrightarrow \mathcal{H}(mD) \longrightarrow \bigoplus_{i=1}^k \mathcal{F}(mD) \longrightarrow \mathcal{Q}(mD) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{K}(mD) \longrightarrow \mathcal{G}(mD) \longrightarrow \mathcal{H}(mD) \longrightarrow 0,$$

where $\mathcal{H}$ is a coherent sheaf. Note that the support of the sheaves $\mathcal{K}$ and $\mathcal{Q}$ is smaller than the support of $\mathcal{F}$. But then by induction any twist of their higher cohomology must vanish, for m sufficiently large. Since the higher cohomology of $f_*\mathcal{G}(mD)$ also vanishes, as f^*D is ample, looking at the second exact sequence, we have that

$$H^i(Y, \mathcal{H}(mD)) = 0,$$

for $i > 0$ and m sufficiently large. Looking at the first exact sequence, it follows that

$$H^i(X, \mathcal{F}(mD)) = 0,$$

for all m sufficiently large. $\square$

Lemma 3.6. *Let $f: X \rightarrow Y$ be a morphism of projective varieties and let D be a $\mathbb{Q}$ -Cartier divisor on Y .*

- (1) *If D is semiample then f^*D is semiample.*
- (2) *If Y is normal, f is surjective and f^*D is semiample then D is semiample.*

Proof. Suppose that D is semiample. Then $|mD|$ is base point free for some $m \in \mathbb{N}$. Pick $x \in X$. By assumption we may find $D' \in |mD|$ not containing $y = f(x)$. But then $f^*D' \in |mf^*D|$ does not contain x . Thus $|mf^*D|$ is base point free and so f^*D is semiample. This proves (1).

Now suppose that Y is normal, f is surjective and f^*D is semiample. Consider the Stein factorisation of $f = g \circ h$, where $g: X \rightarrow Z$ has connected fibres and $h: Z \rightarrow Y$ is a finite morphism of normal varieties. (In fact $Z = \text{Spec}_Y f_*\mathcal{O}_X$ over Y , so that $g_*\mathcal{O}_X = \mathcal{O}_Z$). Thus we only need to prove (2) in the two special cases when f is finite and when f has connected fibres.

First assume that f is a finite morphism of normal varieties. Then we may find a morphism $g: W \rightarrow X$ such that the composition $h = f \circ g: W \rightarrow Y$ is Galois (take W to be the normalisation of X in the Galois closure of $K(X)/K(Y)$). By what we already proved g^*D is semiample. Replacing f by h , we may assume that f is Galois, so that $Y = X/G$ for some finite group G . Pick m so that mf^*D is base point free. Let $y \in Y$ and $x_1, x_2, \dots, x_k \in X$ be the finitely many points lying

over y . Pick $\sigma \in H^0(X, \mathcal{O}_X(mf^*D))$ not vanishing at any one of the points $x_1, x_2, \dots, x_k$. Let $\sigma_1, \sigma_2, \dots, \sigma_l$ be the Galois conjugates of σ . Then each σ_i does not vanish at each x_j and $\prod_{i=1}^l \sigma_i$ is G -invariant, so that it is the pullback of a section $\tau \in H^0(Y, \mathcal{O}_Y(mlD))$ not vanishing at y . But then D is semiample.

Now suppose that the fibres of f are connected. Suppose that f^*D is semiample. Then $|mf^*D|$ is base point free for some $m > 0$. In this case

$$H^0(X, \mathcal{O}_X(mf^*D)) = H^0(Y, \mathcal{O}_Y(mD)).$$

Pick $y \in Y$. Let x be a point of the fibre $X_y = f^{-1}(y)$. Then there is a divisor $D' \in |mf^*D|$ not containing x . But $D' = f^*D''$ where $D'' \in |mD|$. Since D' does not contain x , D'' does not contain y . But then D is semiample. $\square$

Lemma 3.7. *Let X be a projective variety and let D be a $\mathbb{Q}$ -Cartier divisor.*

If H is an ample divisor then there is an integer m_0 such that $D+mH$ is ample for all $m \geq m_0$.

Proof. By induction on the dimension n of X . By (3.4) we may assume that X is normal. It is straightforward to check that a divisor on a curve is ample iff its degree is positive. In particular this result is easy for $n \leq 1$.

So suppose that $n > 1$. Replacing D and H by a multiple we may assume that D is Cartier, so that it is in particular an integral Weil divisor and we may assume that H is very ample. Let $x \neq y \in X$ be any two points of X and let $Y \in |H|$ be a general element containing x and y . Then Y is a projective variety and there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X(D + (m-1)H) \longrightarrow \mathcal{O}_X(D + mH) \longrightarrow \mathcal{O}_Y(D + mH) \longrightarrow 0.$$

By induction on the dimension there is an integer m_0 such that $(D + mH)|_Y$ is ample for all $m \geq m_0$. Pick a positive integer k such that $k(D + mH)|_Y$ is very ample. Note that very ample plus very ample is very ample (just use the Segre embedding). By Serre vanishing, possibly replacing m_0 by a larger integer, we may assume that $k(D + mH)|_Y$ is very ample and that

$$H^i(X, \mathcal{O}_X(k(D + mH))) = 0,$$

for all $m \geq m_0$ and $i > 0$. In particular

$$H^0(X, \mathcal{O}_X(k(D + mH))) \longrightarrow H^0(Y, \mathcal{O}_Y(k(D + mH))),$$

is surjective. By assumption we may find $B \in |k(D + mH)|_Y|$ containing x but not containing y and we may lift this to $B' \in |k(D + mH)|$

containing x by not containing y . It follows that $\phi_{k(D+mH)}: X \rightarrow \mathbb{P}^N$ is an injective morphism, whence it is a finite morphism. But then $k(D + mH)$ is the pullback of an ample divisor under a finite morphism and so it must be ample. $\square$

Theorem 3.8 (Asymptotic Riemann-Roch). *Let X be a normal projective variety and let D and E be two Cartier divisors.*

Then

$$P(m) = \chi(\mathcal{O}_X(mD + E)) = \frac{D^n m^n}{n!} + \frac{D^{n-1} \cdot (K_X - 2E)m^{n-1}}{2(n-1)!} \dots,$$

is a polynomial of degree at most $n = \dim X$, where dots indicate lower order terms.

Proof. By induction on the dimension n of X . Suppose that $n = 1$. Then X is a smooth curve. Riemann-Roch for $mD + E$ then reads

$$\chi(\mathcal{O}_X(mD + E)) = md + e - g + 1 = am - b,$$

where

$$a = d = \frac{\deg D}{1!} \quad \text{and} \quad b = g - 1 - e = \frac{\deg(K_X - 2E)}{2 \cdot 1!}.$$

Now suppose that $n > 1$. Pick a very ample divisor H , which is a general element of the linear system $|H|$, such that $H + D$ is very ample and let $G \in |D + H|$ be a general element. Then G and H are normal projective varieties and there are two exact sequences

$$0 \rightarrow \mathcal{O}_X(mD + E) \rightarrow \mathcal{O}_X(mD + E + H) \rightarrow \mathcal{O}_H(mD + E + H) \rightarrow 0,$$

and

$$0 \rightarrow \mathcal{O}_X((m-1)D + E) \rightarrow \mathcal{O}_X(mD + E + H) \rightarrow \mathcal{O}_G(mD + E + H) \rightarrow 0.$$

Hence

$$\begin{aligned} \chi(X, \mathcal{O}_X(mD + E)) - \chi(X, \mathcal{O}_X(mD + E + H)) &= -\chi(H, \mathcal{O}_H(mD + E + H)) \\ \chi(X, \mathcal{O}_X((m-1)D + E)) - \chi(X, \mathcal{O}_X(mD + E + H)) &= -\chi(G, \mathcal{O}_G(mD + E + H)), \end{aligned}$$

and taking the difference we get

$$\begin{aligned} P(m) - P(m-1) &= \chi(G, \mathcal{O}_G(mD + E + H)) - \chi(H, \mathcal{O}_H(mD + E + H)) \\ &= \frac{(D^{n-1} \cdot G - D^{n-1} \cdot H)m^{n-1}}{(n-1)!} + \dots \\ &= \frac{D^n m^{n-1}}{(n-1)!} + \dots, \end{aligned}$$

is a polynomial of degree $n-1$, by induction on the dimension. The result follows by standard results on the difference polynomial $\Delta P(m) = P(m+1) - P(m)$. $\square$

Theorem 3.9 (Nakai-Moishezon). *Let X be a normal projective variety and let D be a $\mathbb{Q}$ -Cartier divisor.*

The following are equivalent:

- (1) D is ample.
- (2) For every subvariety $V \subset X$ of dimension k ,

$$D^k \cdot V > 0.$$

Proof. Suppose that D is ample. Then mD is very ample for some $m > 0$. Let $\phi: X \rightarrow \mathbb{P}^N$ be the corresponding embedding. Then $mD = \phi^*H$, where H is a hyperplane in $\mathbb{P}^N$. Then

$$m^k D^k \cdot V = H^k \cdot \phi(V) > 0,$$

since intersecting $\phi(V)$ with H^k corresponds to intersecting V with a linear space of dimension $N - k$. But this is nothing more than the degree of $\phi(V)$ in projective space.

Now suppose that D satisfies (2). Let H be a general element of a very ample linear system. Then we have an exact sequence

$$0 \rightarrow \mathcal{O}_X(pD + (q-1)H) \rightarrow \mathcal{O}_X(pD + qH) \rightarrow \mathcal{O}_H(pD + qH) \rightarrow 0.$$

By induction, $D|_H$ is ample. Thus

$$h^i(H, \mathcal{O}_H(pD + qH)) = 0,$$

for $i > 0$, p sufficiently large and any $q \geq 0$. In particular,

$$h^i(X, \mathcal{O}_X(pD + (q-1)H)) = h^i(X, \mathcal{O}_X(pD + qH)),$$

for $i > 1$, p sufficiently large and any $q \geq 1$. By Serre vanishing the last group vanishes for q sufficiently large. Thus by descending induction

$$h^i(X, \mathcal{O}_X(pD + qH)) = 0,$$

for all $q \geq 0$. Thus by (3.8) it follows that

$$h^0(X, \mathcal{O}_X(mD)) \neq 0,$$

for m sufficiently large, that is $|mD|$ is non-empty. As usual, this means that we may assume that $D \geq 0$ is Cartier. Let $\nu: \tilde{D} \rightarrow D_{\text{red}}$ be the normalisation of D_{red} , the reduced subscheme associated to D . Then $\nu^*D|_{D_{\text{red}}}$ is ample by induction. It follows by (3.4) that $D|_D$ is ample.

I claim that the map

$$\rho_m: H^0(X, \mathcal{O}_X(mD)) \rightarrow H^0(D, \mathcal{O}_D(mD)),$$

is surjective for m sufficiently large. Consider the exact sequence

$$0 \rightarrow \mathcal{O}_X((m-1)D) \rightarrow \mathcal{O}_X(mD) \rightarrow \mathcal{O}_D(mD) \rightarrow 0,$$

As $D|_D$ is ample,

$$h^i(D, \mathcal{O}_D(mD)) = 0,$$

for $i > 0$ and m sufficiently large, by Serre vanishing. Thus

$$h^1(X, \mathcal{O}_X(mD)) \leq h^1(X, \mathcal{O}_X((m-1)D)) \quad \text{and} \quad h^i(X, \mathcal{O}_X(mD)) = 0,$$

for $i > 1$, with equality if and only if ρ_m is surjective. Since

$$h^1(X, \mathcal{O}_X(mD)),$$

is finite dimensional, its dimension cannot drop infinitely often, and so ρ_m is surjective as claimed.

As $D|_D$ is ample, $(mD)|_D$ is very ample. As we can lift sections, it follows that $|mD|$ is base point free, that is D is semiample. Let $\phi = \phi_{mD}: X \rightarrow \mathbb{P}^N$ be the corresponding morphism. Then $D = \phi^*H$. Suppose that C is a curve contracted by ϕ . Then

$$D \cdot C = \phi^*H \cdot C = H \cdot \phi_*C = 0,$$

a contradiction. But then ϕ_{mD} is a finite morphism and $D = \phi^*H$ is ample by (3.4). $\square$

A very natural question to ask with respect to both very ampleness and vanishing, is if there are effective versions. In general the answer is definitely no, but both questions have a (partially conjectural) very sharp answer:

Theorem 3.10 (Kodaira vanishing). *Let X be a smooth projective variety and let D be an ample divisor. Then*

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0,$$

for $i > 0$.

Kodaira vanishing is one of the most fundamental and important results in higher dimensional geometry. To prove it one needs to use some analytic methods (some Hodge theory and positivity of various metrics) or some very deep results of Deligne and Illusie on Hodge theory in characteristic p . In fact Kodaira vanishing fails in general in characteristic p .

Using Kodaira vanishing one can prove a technically more powerful result, Kawamata-Viehweg vanishing, whose proof is purely algebraic and whose form dictates most of the definitions and shape of the whole subject of higher dimensional geometry.

Conjecture 3.11 (Fujita's conjecture). *Let X be a smooth projective variety of dimension n and let D be an ample divisor.*

Then

- (1) $K_X + (n+1)D$ is base point free.
- (2) $K_X + (n+2)D$ is very ample.

It is easy to give examples which show that Fujita's conjecture is sharp:

Example 3.12. *Let $X = \mathbb{P}^n$ and $D = H$. Then*

$$K_X + dH \sim (d - n - 1)H.$$

Thus $K_X + dH$ is base point free if and only if $d \geq n + 1$ and it is ample if and only if $d \geq n + 2$.

Example 3.13. *Let C be a smooth curve and let $D = p$. Then $K_C + p$ is never base point free.*

Indeed I claim that p is a base point of $|K_C + p|$. To see this consider

$$|K_C| \longrightarrow |K_C + p| \quad \text{given by} \quad D \longrightarrow D + p.$$

This map is linear and injective. By Riemann-Roch

$$\dim |K_C| = (2g - 2 - g + 1) - 1 + h^1(C, \mathcal{O}_C(K_C)) = g - 2 + h^0(C, \mathcal{O}_C) = g - 1$$

$$\dim |K_C + p| = (2g - 1 - g + 1) - 1 + h^1(C, \mathcal{O}_C(K_C + p)) = g - 1 + h^0(C, \mathcal{O}_{C(-p)}) = g - 1.$$

Thus the map is surjective and

$$|K_C + p| = |K_C| + p.$$

But then p is a base point of the linear system $|K_C + p|$. On the other hand, similar calculations show that $|K_C + p + q|$ is a free linear system, but that $\phi_{K_C + p + q}$ is not injective if $p \neq q$ and not an embedding if $p = q$, so that $K_C + p + q$ is never very ample.

Here is a baby version of (3.11):

Lemma 3.14. *Let X be a smooth projective variety.*

If D is a very ample divisor then $K_X + (n + 1)D$ is free, and $K_X + (n + 2)D$ is very ample.

Proof. We first show (1). We proceed by induction on the dimension of X . Let $x \in X$ be a point of X and let $Y \in |D|$ be a general element containing $x \in X$. Then Y is smooth and

$$(K_X + (n + 1)D)|_Y \sim (K_X + Y + nD)|_Y = K_Y + nD|_Y.$$

By induction $K_Y + nD|_Y$ is free. Thus we may find $D' \in |K_Y + nD|_Y|$ not containing x . There is a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(K_X + nD) \longrightarrow \mathcal{O}_X(K_X + (n + 1)D) \longrightarrow \mathcal{O}_Y(K_Y + nD) \longrightarrow 0.$$

The obstructions to surjectivity of the lifting map

$$H^0(X, \mathcal{O}_X(K_X + (n + 1)D)) \longrightarrow H^0(Y, \mathcal{O}_Y(K_Y + nD)),$$

live in

$$H^1(X, \mathcal{O}_X(K_X + nD)) = 0,$$

which vanishes by Kodaira vanishing. Thus D' lifts to an element D'' of $|K_X + (n+1)D|$ not containing x . Hence $|K_X + (n+1)D|$ is free.

It is not hard to check that (free) + (very ample) is very ample. Thus $K_X + (n+2)D$ is very ample. $\square$

(3.11) is known in dimension two (Reider's Theorem) and freeness is known in dimensions three (Ein-Lazarsfeld) and four (Kawamata). The best general results are due to Anghern and Siu:

Theorem 3.15. *Let X be a smooth projective variety and let D be an ample divisor.*

- (1) $K_X + mD$ is free for $m > \binom{n+1}{2}$, and
- (2) $K_X + mD$ separates points, for $m \geq \binom{n+2}{2}$.

Even to show that $K_X + 5D$ is very ample on a smooth threefold X seems very hard at the moment.

We return to the problem of determining when a line bundle is ample.

Definition 3.16. *Let X be a normal projective variety and let D be a $\mathbb{Q}$ -Cartier divisor. We say that D is **nef** if $D \cdot C \geq 0$ for all curves $C \subset X$.*

Lemma 3.17. *Let X be a normal variety and let D be a $\mathbb{Q}$ -divisor.*

If D is semiample then D is nef.

Proof. Note that D is automatically a $\mathbb{Q}$ -Cartier divisor. By assumption there is a morphism $\phi: X \rightarrow Y \subset \mathbb{P}^n$ such that

$$mD = \phi^*H.$$

But then

$$D \cdot C = \frac{1}{m} \phi_* C \cdot H \geq 0. \quad \square$$

Lemma 3.18. *Let X be a normal projective variety and let D be a nef divisor.*

If $V \subset X$ is any subvariety of X then

$$D^k \cdot V \geq 0,$$

where V has dimension k .

Proof. By induction on $n = \dim X$ it suffices to prove that

$$D^n \geq 0.$$

Pick an ample divisor H . Then $D + tH$ is nef for all $t \geq 0$. We have

$$f(t) = (D + tH)^n = \sum \binom{n}{i} D^i H^{n-i} t^{n-i},$$

is a polynomial in t , all of whose terms are non-negative, except maybe the constant term and which tends to infinity as t tends to infinity. Suppose that $D^n \leq 0$. Then there is a real number $t_0 \in [0, \infty)$ such that

$$f(t_0) = 0,$$

and $f(t) > 0$ for all $t > t_0$. Pick $t > t_0$ rational. Then $D + tH$ is ample, by Nakai's criteria. We may write

$$f(t) = t^n H^n + \sum t^i D H^i (H + tD)^{n-i-1}.$$

Consider the product $H^i (H + tD)^{n-i-1}$. Pick k such that kH is very ample and pick l such that $l(H + tD)$ is very ample. Pick general elements $H_1, H_2, \dots, H_i \in |kH|$ and $G_1, G_2, \dots, G_{n-i-1} \in |l(H + tD)|$. Then the intersection product

$$H_1 \cdot H_2 \cdots H_i \cdot G_1 \cdot G_2 \cdots G_{n-i-1} = k^i l^{n-i-1} H^i \cdot (H + tD)^{n-i-1},$$

is represented by a smooth curve

$$C = \left(\bigcap_a H_a \right) \cap \left(\bigcap G_b \right).$$

Thus every term is non-negative, as $D \cdot C \geq 0$. But then

$$\begin{aligned} 0 &= f(t_0) \\ &= \lim_{t \rightarrow t_0} f(t) \\ &\geq t_0^n H^n. \end{aligned}$$

Thus $t_0 = 0$, and $D^n \geq 0$. □

Lemma 3.19. *Let X be a normal projective variety and let $\pi: Y \rightarrow X$ blow up a smooth point p of X .*

Then $E^n = (-1)^{n-1}$.

Proof. Since this result is local in the analytic topology, we may as well assume that $X = \mathbb{P}^n$. Choose coordinates $x_1, x_2, \dots, x_n$ about the point p . Then coordinates on $Y \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$ are given by the equations

$$x_i Y_j = x_j Y_i.$$

(These equations simply express the fact that $(x_1, x_2, \dots, x_n)$ defines the point $[Y_1 : Y_2 : \dots : Y_n] \in \mathbb{P}^{n-1}$.) On the coordinate chart $Y_n \neq 0$, we have affine coordinates $y_i = Y_i/Y_n$ on $\mathbb{P}^{n-1}$ and since

$$x_i = x_n y_i,$$

it follows that $x_n, y_1, y_2, \dots, y_{n-1}$ are coordinates on Y , and the exceptional divisor is given locally by $x_n = 0$. Let H be the class of a

hyperplane in $\mathbb{P}^n$ which passes through p . Then we may assume that H is given by $x_1 = 0$. Since $x_1 = x_n y_1$ it follows that

$$\pi^* H = G + E,$$

where G defined by $y_1 = 0$, is the strict transform of H . Now $G|_E$ restricts to a hyperplane in E . Thus

$$E|_E = -G|_E,$$

since E pushes forward to zero. But then

$$E^n = (E|_E)^{n-1} = (-1)^{n-1}. \quad \square$$

Definition-Lemma 3.20 (Kodaira's Lemma). *Let X be a normal projective variety of dimension n and let D be a $\mathbb{Q}$ -Cartier divisor.*

The following are equivalent:

(1) $h^0(X, \mathcal{O}_X(mD)) > \alpha m^n$, for some constant $\alpha > 0$, for any m which is sufficiently divisible.

(2) $D \sim_{\mathbb{Q}} A + E$, where A is an ample divisor and $E \geq 0$.

If further D is nef then these conditions are equivalent to

(3) $D^n > 0$.

*If any of these equivalent conditions hold then we say that D is **big**.*

Proof. Let $H \geq 0$ be any ample Cartier divisor. If m is sufficiently large, then

$$h^i(X, \mathcal{O}_X(mH)) = 0,$$

so that by Asymptotic Riemann Roch there are positive constants α_i such that

$$\alpha_1 m^n < h^0(X, \mathcal{O}_X(mH)) < \alpha_2 m^n,$$

for all m . Now let G be any divisor. Pick $k > 0$ such that $G + kH$ is ample. Then

$$h^0(X, \mathcal{O}_X(mG)) \leq h^0(X, \mathcal{O}_X(mG + mkH)) \leq \beta_1 m^n,$$

for some constant β_1 .

Suppose that (1) holds. Let H be an ample Cartier divisor. Then there is an exact sequence

$$0 \longrightarrow \mathcal{O}_X(mD - H) \longrightarrow \mathcal{O}_X(mD) \longrightarrow \mathcal{O}_H(mD) \longrightarrow 0.$$

Now

$$h^0(H, \mathcal{O}_H(mD)) \leq \beta m^{n-1},$$

for some constant β . It follows that

$$h^0(X, \mathcal{O}_X(mD - H)) > \alpha m^n.$$

In particular we may find B such that

$$B \in |mD - H|.$$

But then

$$D \sim_{\mathbb{Q}} H/m + B/m = A + E.$$

Thus (1) implies (2).

Now suppose that (2) holds. Replacing D by a multiple, we may assume that $D \sim A + E$. But then

$$h^0(X, \mathcal{O}_X(mD)) \geq h^0(X, \mathcal{O}_X(mA)) > \alpha m^n,$$

for some constant $\alpha > 0$. Thus (2) implies (1).

Now suppose that D is nef. Assume that (2) holds. We may assume that A is very ample and a general element of $|A|$. Then

$$\begin{aligned} D^n &= A \cdot D^{n-1} + E \cdot D^{n-1} \\ &\geq (D|_A)^{n-1} > 0, \end{aligned}$$

by induction on the dimension. Thus (2) implies (3).

Finally suppose that (3) holds. Let $\pi: Y \rightarrow X$ be a birational morphism such that Y is smooth. Since $G = \pi^*D$ is nef, $G^n = D^n$ and

$$h^0(Y, \mathcal{O}_Y(mG)) = h^0(X, \mathcal{O}_X(mD)),$$

replacing X by Y and D by G , we may assume that X is smooth. Pick a very ample divisor H , a general element of $|H|$, such that $H - K_X$ is also very ample and let $G \in |H - K_X|$ be a general element. There is an exact sequence

$$0 \rightarrow \mathcal{O}_X(mD) \rightarrow \mathcal{O}_X(mD + G) \rightarrow \mathcal{O}_G(mD + G) \rightarrow 0.$$

Now

$$h^i(X, \mathcal{O}_X(mD + G)) = h^i(X, \mathcal{O}_X(K_X + H + mD)) = 0$$

for $i > 0$ and $m \geq 0$ by Kodaira vanishing, as $H + mD$ is ample. Thus

$$h^0(X, \mathcal{O}_X(mD + G)) > \alpha m^n,$$

for some constant $\alpha > 0$, by Asymptotic Riemann-Roch. Since

$$h^0(G, \mathcal{O}_G(mD + G)) < \beta m^{n-1},$$

for some constant β , (3) implies (1). $\square$

Theorem 3.21 (Seshadri's criteria). *Let X be a normal projective variety and let D be a $\mathbb{Q}$ -Cartier divisor.*

The following are equivalent:

(1) D is ample.

- (2) For every point $x \in X$, there is a non-negative constant $\epsilon = \epsilon(x) > 0$ such that for every curve C ,

$$D \cdot C > \epsilon \operatorname{mult}_x C,$$

where $\operatorname{mult}_x C$ is the multiplicity of the point x on C .

Proof. Suppose that D is ample. Then mD is very ample for some positive integer m . Let C be a curve with a point x of multiplicity k . Pick y any other point of C . Then we may find $H \in |mD|$ containing x and not containing y . In particular H does not contain C . In this case

$$\begin{aligned} (mD) \cdot C &= H \cdot C \\ &\geq k, \end{aligned}$$

so that $\epsilon = 1/m$ will do. Thus (1) implies (2).

Now assume that (2) holds. We check the hypotheses for Nakai's criteria. By induction on the dimension n of X it suffices to check that $D^n > 0$. Let $\pi: Y \rightarrow X$ be the blow up of X at x , a smooth point of X , with exceptional divisor E . Consider $\pi^*D - \eta E$, for any $0 < \eta < \epsilon$. Let $\Sigma \subset Y$ be any curve on Y . If Σ is contained in E , then

$$(\pi^*D - \eta E) \cdot \Sigma = -E \cdot \Sigma > 0.$$

Otherwise let C be the image of Σ . If the multiplicity of C at x is m , then $E \cdot \Sigma = m$. Thus

$$(\pi^*D - \eta E) \cdot \Sigma = D \cdot C - \eta m > 0,$$

by definition of ϵ . It follows that $\pi^*D - \eta E$ is nef and so $\pi^*D - \epsilon E$ is nef. By (3.18) it follows that the polynomial

$$f(t) = (\pi^*D - tE)^n,$$

of degree n in t is non-negative. On the other hand, note that $E^n = \pm 1 \neq 0$. Thus the polynomial $f(t)$ is not constant. Thus $f(\eta) > 0$, some $0 < \eta < \epsilon$. It follows that

$$h^0(X, \mathcal{O}_X(mD)) \geq h^0(Y, \mathcal{O}_X(m\pi^*D - m\eta E)) > \alpha m^n,$$

for m sufficiently large and divisible for some m . It follows that $D^n > 0$. $\square$

One of the most interesting aspects of Seshadri's criteria is that gives a local measure of ampleness:

Definition 3.22. Let X be a normal variety, and let D be a nef $\mathbb{Q}$ -Cartier divisor. Given a point $x \in X$ in the smooth locus of X , let $\pi: Y \rightarrow X$ be the blow up of X at x . The real number

$$\epsilon(D, x) = \sup\{\epsilon \mid \pi^*D - \epsilon E \text{ is nef}\}$$

is called the **Seshadri constant** of D at x .

It seems to be next to impossible to calculate the Seshadri constant in any interesting cases. For example there is no known example of a smooth surface S and a point $x \in S$ such that the Seshadri constant is irrational, although this is conjectured to happen nearly all the time. One of the first interesting cases is a very general smooth quintic surface S in $\mathbb{P}^3$ (so that S belongs to the complement of a countable union of closed subsets of the space of all quintics $\mathbb{P}^{55}$). Suppose that $p \in S$ is a very general point. Let $\pi: T \rightarrow S$ blow up the point p . As S is very general,

$$\text{Pic}(T) = \mathbb{Z}[\pi^*H] \oplus \mathbb{Z}[E],$$

where H is the class of a hyperplane and E is the exceptional divisor. Since p is very general, it seems reasonable to expect that the only curve of negative self-intersection on T is E . If this is the case then $\pi^*H - aE$ is nef if and only if its self-intersection is non-negative. Now

$$\begin{aligned} 0 &= (\pi^*H - aE)^2 \\ &= H^2 - a^2 \\ &= 5 - a^2. \end{aligned}$$

So if there are no curves of negative self-intersection other than E , then the Seshadri constant is $\sqrt{5}$.

Finally let us turn to one of the most interesting ampleness criteria.

Definition 3.23. Let X be a normal projective variety. The **cone of curves** of X , denoted $\text{NE}(X)$, is the cone spanned by the classes $[C] \in H_2(X, \mathbb{R})$ inside the vector space $H_2(X, \mathbb{R})$. The **closed cone of curves** of X , denoted $\overline{\text{NE}}(X)$, is the closure of $\text{NE}(X)$ inside $H_2(X, \mathbb{R})$.

Note that a Cartier D defines a linear functional on $H_2(X, \mathbb{R})$. Indeed we may consider D as a cohomology class $c_1(D) \in H^2(X, \mathbb{R})$ and we get a number by taking the cap product. If C is a curve then we have

$$D \cdot C = c_1(D) \cap [C].$$

The significance of the closed cone of curves is given by:

Theorem 3.24 (Kleiman's criterion). Let X be a normal projective variety.

The following are equivalent:

- (1) D is ample.
- (2) The linear functional defined by D is positive on

$$\overline{\text{NE}}(X) - \{0\} \rightarrow \mathbb{R}^+$$

In particular $\overline{\text{NE}}(X)$ does not contain a line and if H is ample then the set

$$\{\alpha \in \overline{\text{NE}}(X) \mid H \cdot \alpha \leq k\},$$

is compact, where k is any positive constant.

Proof. Suppose that D is ample. Then D is certainly positive on $\text{NE}(X) - \{0\}$. Suppose that it is not positive on $\overline{\text{NE}}(X) - \{0\}$. Then there is a non-zero class $\alpha \in \overline{\text{NE}}(X)$ such that $D \cdot \alpha = 0$, which is a limit of curve classes α_i , where

$$\alpha_i = \sum a_{ij} [C_{ij}]$$

where $a_{ij} > 0$ are positive real numbers. Pick a $\mathbb{Q}$ -Cartier divisor M such that $M \cdot \alpha < 0$. Then there is a positive integer m such that $mH + M$ is ample. But then

$$0 > (mH + M) \cdot \alpha = \lim_i (mH + M) \cdot \alpha_i \geq 0,$$

a contradiction. Thus (1) implies (2).

Suppose that $\overline{\text{NE}}(X)$ contains a line. Then it contains a line through the origin, which is impossible, since X contains an ample divisor H , which is positive on this line outside the origin. Pick a basis $M_1, M_2, \dots, M_k$ for the Néron-Severi group $\text{NS}(X)$ of X , the group of Weil divisors modulo numerical equivalence, which is the vector space dual to the space spanned by $\overline{\text{NE}}(X)$. Then we may pick m such that $mH \pm M_i$ is ample for all $1 \leq i \leq k$. In particular

$$|M_i \cdot \alpha| \leq mH \cdot \alpha.$$

Thus the set above is compact as it is a closed subset of a big cube.

Now suppose that (2) holds. By what we just proved, the set of $\alpha \in \overline{\text{NE}}(X)$ such that $H \cdot \alpha = 1$ forms a compact slice of $\overline{\text{NE}}(X)$ (meaning that this set is compact and $\overline{\text{NE}}(X)$ is the cone over this convex set). Thus $B = D - \epsilon H$ is nef, for some $\epsilon > 0$. But then

$$D = (D - \epsilon H) + \epsilon H = B + \epsilon H,$$

is ample, by (3.18) and Nakai's criterion. $\square$

Mori realised that Kleiman's criteria presents a straightforward way to classify projective varieties.

Definition 3.25. Let $f: X \rightarrow Y$ be a morphism of varieties. We say that f is a **contraction** morphism if $f_*\mathcal{O}_X = \mathcal{O}_Y$.

A contraction morphism always has connected fibres; if Y is normal and f has connected fibres then f is a contraction morphism. Consider the category of projective varieties and contraction morphisms. Any

morphism $f: X \longrightarrow Y$ is determined by the curves contracted by f . Indeed let $\sim$ be the equivalence relation on the points of X , generated by declaring two points to be equivalent if they both belong to an irreducible curve contracted by f . Thus $x \sim y$ if and only if x and y belong to a connected curve C which is contracted to a point.

I claim that $x \sim y$ determines f . First observe that $x \sim y$ if and only if $f(x) = f(y)$. It is clear that if $x \sim y$ then $f(x) = f(y)$. Conversely suppose that $f(x) = f(y) = p$. If $d = \dim f^{-1}(p) \leq 1$ there is nothing to prove. If $d > 1$ then since the fibres of f are connected, we may assume that x and y belong to the same irreducible component. Pick an ample divisor $H \subset f^{-1}(p)$ containing both x and y . Then $\dim H = d - 1$. Repeating this operation we find a curve containing both x and y . This determines $Y = X/\sim$ as a topological space, and the condition $\mathcal{O}_Y = f_*\mathcal{O}_X$ determines the scheme structure.

Note also that there is then a partial correspondence

- (1) faces of the Mori cone $\overline{\text{NE}}(X)$.
- (2) contraction morphisms $\phi: X \longrightarrow Y$

Given a contraction morphism ϕ , let

$$F = \{ \alpha \in \overline{\text{NE}}(X) \mid D \cdot \alpha = 0 \},$$

where $D = \phi^*H$. Kleimans' criteria then says that F is a face. The problem is that given a face F of the closed cone it is in general impossible to contract F . For a start F must be rational (that is spanned by integral classes).

Equivalently and dually, the problem is that there are nef divisors which are not semiample.