

## FOURTH HOMEWORK

1. Let  $C$  be a smooth curve of genus 2 over  $\mathbb{C}$ . Let

$$C_2 = \frac{C \times C}{\langle \sigma \rangle},$$

by the symmetric product of  $C$  with itself, where  $\sigma$  is the natural involution switching the factors.

- (i) Show that  $C_2$  is a smooth surface.
- (ii) Show that points of  $C_2$  correspond to divisors of degree two.
- (iii) If  $\pi: C_2 \rightarrow \text{Pic}^2 C$  is the natural map (that is, the Abel-Jacobi map) which sends a divisor  $p + q$  of degree two to the corresponding line bundle  $\mathcal{O}_C(p + q)$  of degree two, then show that  $\pi$  collapses a copy of  $\mathbb{P}^1$  to a point (use the fact that  $\text{Pic}^2 C$  is isomorphic to the Jacobian, an abelian surface).
- (iv) If  $C$  is generic, then the Picard number of  $C_2$  is two, and the class of the image of a fibre  $x$  and the diagonal  $\delta$  span the Néron-Severi group. Identify  $\overline{\text{NE}}(C_2)$  in this case.
- (v) What is the Seshadri constant of a principally polarised abelian surface of Picard number one (you may use the fact that any such arises as the Jacobian of a curve of genus 2).

2. Let  $V$  be a finite dimensional vector space over an algebraically closed field of characteristic  $p > 0$ . Suppose that  $f: V \rightarrow V$  is a  $q$ -linear map, so that  $f$  is additive,  $f(w + v) = f(w) + f(v)$  and  $f(\lambda v) = \lambda^q v$  for all vectors  $w$  and  $v$  and scalars  $\lambda$ . Show that we may decompose  $V$  as  $V_n \oplus V_s$  which respects the map  $f$ , where the restriction of  $f$  to  $V_n$  is nilpotent and there is a basis  $v_1, v_2, \dots, v_k$  of  $V_s$ , such that  $f(v_i) = v_i$ .

3. Let  $X$  be a normal projective variety over an algebraically closed field of characteristic  $p$ . Show that there is a natural correspondence between cyclic covers of degree  $p$  and elements  $\alpha$  of  $H^1(X, \mathcal{O}_X)$  which are invariant under Frobenius (*use Čech cohomology to write out explicitly what it means for  $F^*\alpha = \alpha$  in cohomology. You will probably need to remind yourself what a cyclic extension of order  $p$  in characteristic  $p$  looks like.*).

4. Let  $S$  be a smooth surface and let  $C$  be a smooth irreducible curve in  $S$  over a finite field. If  $C$  is nef and  $g(C) \leq 1$  then we are going to prove that  $C$  is semiample.

- (i) Show that if  $C^2 > 0$  then  $C$  is semiample.
- (ii) If  $C^2 = 0$  then show that  $C|_C$  is torsion.

(iii) If  $m$  is the smallest positive integer such that  $mC|_C \sim 0$  then show that

$$H^1(S, \mathcal{O}_C(kC)) \simeq H^1(S, \mathcal{O}_S),$$

for all  $0 \leq k < m$ .

(iv) Show that we can evaporate  $H^1(S, \mathcal{O}_S(kC))$  and conclude that we can lift sections from an appropriate power of  $\mathcal{O}_C(mC)$ .