TAKE HOME FINAL EXAM
MATH 120A, UCSD, WINTER 20

You have 24 hours.

There are 5 problems, and the total number of points is 100.

Please make your work as clear and easy to follow as possible. There is no need to be verbose but explain all of the steps, using your own words. You may consult the lecture notes and model answers but you may not use any other reference nor may you confer with anyone. You may use any of the standard results in the lecture notes as long as you clearly state what you are using. If you don't know how to solve the whole problem answer the portion you can solve.

Please submit your answers on Gradescope by 11:30am on Thursday March 19th.

1. (20pts) (a) If the power series $\sum_n a_n(z-1)^n$ converges at $2i$ will it converge at $-i$?
(b) Suppose that a function $f(z)$ has three isolated singularities at a_1, a_2 and a_3 and is otherwise holomorphic on the whole complex plane. Consider how many Laurent expansions there are with centre a.
(i) Give an example with the maximum number of Laurent expansions (that is, write down an explicit function $f(z)$, the location of the isolated singularities a_1, a_2, a_3 and the centre of the expansion a. The simpler the example is the better. No need to give the expansions).
(ii) Give an example with the minimum number.

2. (20pts) Suppose that $f(z)$ is an entire function with no zeroes on any of the circles $|z| = n$, $n = 1, 2, 3, \ldots$, and yet

$$\oint_{|z|=n} \frac{\mathrm{d}z}{f(z)} \neq \oint_{|z|=n+1} \frac{\mathrm{d}z}{f(z)}.$$

What can you say about the zeroes of $f(z)$? Conclude that $f(z)$ is not a polynomial.

3. (20pts) Show that the unit disk Δ is conformally equivalent to $\Delta \cap \mathbb{H}$ the portion of the unit disk above the real axis, that is, find a holomorphic bijection between Δ and $\Delta \cap \mathbb{H}$.

4. (20pts) If $f(z)$ is an entire function, ω is a complex number and $a < b$ are real numbers such that

$$f((r+1)\omega) = f(r\omega) \qquad \text{where} \qquad a < r < b,$$

conclude that

$$f(z+\omega) = f(z)$$

for all complex numbers z, that is, show that $f(z)$ is periodic with period ω.

5. (20pts) If you know the values of a holomorphic function $f(z)$ on the boundary of a region U, which is piecewise differentiable, how would you estimate the absolute value of its nth derivative inside U?

6. (Extra credit: 10pts) If the power series $\sum_n a_n z^n$ converges at i will it converge at $-i$?
7. (Extra credit: 20pts) Let $f(z)$ be a function with finitely many isolated singularities which is holomorphic everywhere else on $\mathbb{C}$. Suppose that

$$|f(z)| \leq \left(\frac{|z|}{|z+1|}\right)^{3/2}$$

What can you conclude about $f(z)$?