TAKE HOME FINAL EXAM
MATH 120A, UCSD, WINTER 20

You have 24 hours.

There are 5 problems, and the total number of points is 100.

Please make your work as clear and easy to follow as possible. There is no need to be verbose but explain all of the steps, using your own words. You may consult the lecture notes and model answers but you may not use any other reference nor may you confer with anyone. You may use any of the standard results in the lecture notes as long as you clearly state what you are using. If you don't know how to solve the whole problem answer the portion you can solve.

Please submit your answers on Gradescope by 11:30am on Thursday March 19th.

1. (20pts) (a) This power series is centred at 1. It converges at $2i$ so that it converges on the open disk centred at 1 containing $2i$ in the boundary.
$-i$ belongs to this disk and so it converges at $-i$.
(b)
(i) Consider the function

$$f(z) = \frac{1}{z-1} + \frac{1}{z-2} + \frac{1}{z-3}.$$

It has isolated singularities at $a_1 = 1$, $a_2 = 2$ and $a_3 = 3$. Suppose that we centre our expansion at zero. There are three relevant circles. All circles are centred at 0. The first circle has radius one and contains a_1. The second circle has radius two and contains a_2. The third circle has radius three and contains a_3.
There are four relevant annuli, U_0, U_1, U_2 and U_3, the regions between two consecutive circles:

$$\begin{aligned} U_0 &= \{\, z \in \mathbb{C} \,|\, |z| < 1 \,\} \\ U_1 &= \{\, z \in \mathbb{C} \,|\, 1 < |z| < 2 \,\} \\ U_2 &= \{\, z \in \mathbb{C} \,|\, 2 < |z| < 3 \,\} \\ U_3 &= \{\, z \in \mathbb{C} \,|\, 3 < |z| \,\}. \end{aligned}$$

Accordingly there are four different Laurent expansions. It is clear that this is the maximum.
(ii) Consider the function

$$f(z) = \frac{1}{z-1} + \frac{1}{z-i} + \frac{1}{z+1}.$$

It has singularities at $a_1 = 1$, $a_2 = i$ and $a_3 = -1$. Suppose that we centre our expansion at zero. There is one relevant circle. The unit circle contains all three points a_1, a_2 and a_3.
There are two relevant annuli, U_0 and U_1, the regions:

$$\begin{aligned} U_0 &= \{\, z \in \mathbb{C} \,|\, |z| < 1 \,\} \\ U_1 &= \{\, z \in \mathbb{C} \,|\, 1 < |z| \,\}. \end{aligned}$$

Accordingly there are two different Laurent expansions. It is clear that this is the minimum.

2. (20pts) As $f(z)$ has no zeroes on any of the circles it is not identically zero. In particular the zeroes of $f(z)$ are isolated. It follows that $f(z)$ has finitely many zeroes in any circle.
The singularities of
$$\frac{1}{f(z)}$$
are located at the zeroes of $f(z)$. Thus
$$\frac{1}{f(z)}$$
has finitely many isolated singularities in any circle.
The residue theorem states that
$$\oint_{|z|=n} \frac{\mathrm{d}z}{f(z)} = 2\pi i \sum_a \mathrm{Res}_a \frac{1}{f(z)},$$
where the sum ranges over the isolated singularities of
$$\frac{1}{f(z)}$$
inside the open disk of radius n centred at 0. If
$$\oint_{|z|=n} \frac{\mathrm{d}z}{f(z)} \neq \oint_{|z|=n+1} \frac{\mathrm{d}z}{f(z)}.$$
then there must be more isolated singularities inside the disk of radius $n+1$ than inside the disk of radius n. It follows that there are more zeroes of $f(z)$ inside the disk of radius $n+1$ than inside the disk of radius n.
In particular $f(z)$ has infinitely many zeroes. It follows that $f(z)$ is not a polynomial.

3. (20pts) We do this in stages. The region $\Delta \cap \mathbb{H}$ is bounded by a line and a circle. We use a Möbius transformation to send one of the intersection points 1 to infinity,

$$z \longrightarrow \frac{1}{1-z}.$$

This fixes the real line. It sends the other point of intersection -1 to $1/2$. The circle gets sent to a line through $1/2$. The unit circle intersects the real line at a right angle. As Möbius transformations are conformal the circle gets sent to the vertical line through $1/2$, the line $\mathrm{Re}(z) = 1/2$.

The interval $[-1, 1]$ gets sent to the interval $[1/2, \infty]$. The point i is sent to $1/2 + i/2$, so the semicircle, $|z| = 1$ and $\mathrm{Im}(z) > 0$ gets sent to the half line $\mathrm{Re}(z) = 1/2$, $\mathrm{Im}(z) > 0$.

The point $i/2$ gets sent to

$$\begin{aligned} \frac{1}{1-i/2} &= \frac{1+i/2}{1+1/4} \\ &= \frac{1+i/2}{5/4} \\ &= \frac{4}{5} + \frac{2i}{5}. \end{aligned}$$

This implies that the region is sent to the quadrant

$$\{\, z \in \mathbb{C} \mid \mathrm{Im}(z) > 0, \mathrm{Re}(z) > 1/2 \,\}.$$

If we translate by $-1/2$, $z \longrightarrow z - 1/2$, we move the region to the first quadrant,

$$\{\, z \in \mathbb{C} \mid \mathrm{Im}(z) > 0, \mathrm{Re}(z) > 0 \,\}.$$

If we square $z \longrightarrow z^2$ we expand the first quadrant to the upper half plane $\mathbb{H}$.

Finally, it is shown in question 7 of hwk 2 that

$$z \longrightarrow \frac{-2iz + 1 - i}{-2z - 1 + i},$$

takes the upper half plane $\mathbb{H}$ to the unit circle Δ.

4. (20pts) Let $g(z) = f(z+\omega) - f(z)$. Then $g(z)$ is an entire function. If z is a point on the line segment connecting $a\omega$ to $b\omega$ then $z = r\omega$ for some $a < r < b$. It follows that

$$\begin{aligned} g(z) &= g(r\omega) \\ &= f(r\omega + \omega) - f(r\omega) \\ &= f((r+1)\omega) - f(r\omega) \\ &= 0. \end{aligned}$$

Thus $g(z)$ is zero on the line segment connecting $a\omega$ to $b\omega$.
As the zeroes of $g(z)$ are not isolated it follows that $g(z)$ is identically zero, so that $g(z) = 0$ for all z. But then

$$f(z+\omega) = f(z)$$

for all complex numbers z, that is, $f(z)$ is periodic with period ω.

5. (20pts) We use Cauchy's formula:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_{\partial U} \frac{f(z)}{(z-a)^{n+1}} \mathrm{d}z.$$

To estimate the absolute value of the nth derivative, we just need to estimate the absolute value of the RHS. For this we need the length L of the boundary and the maximum value M of the absolute value of the integrand. To estimate M, suppose that M_0 is the maximum value of $|f(z)|$ on the boundary and r is the minimum distance of a to a point on the boundary. We have

$$\begin{aligned} \left| \frac{f(z)}{(z-a)^{n+1}} \right| &= \frac{|f(z)|}{|z-a|^{n+1}} \\ &\leq \frac{M_0}{r^{n+1}}. \end{aligned}$$

In this case

$$\begin{aligned} |f^{(n)}(a)| &= \left| \frac{n!}{2\pi i} \int_{\partial U} \frac{f(z)}{(z-a)^{n+1}} \mathrm{d}z \right| \\ &\leq \frac{n!}{2\pi} L \frac{M_0}{r^{n+1}}. \end{aligned}$$

6. No.
Consider

$$f(z) = \mathrm{Log}(1-z).$$

This is holomorphic inside the disk of radius 1 about the origin. Therefore it has a power series expansion centred around the origin. If we differentiate then we get

$$-\frac{1}{1-z}.$$

The power series for this is

$$-\frac{1}{1-z} = -1 - z - z^2 - z^3 - \dots.$$

If we integrate this term by term we get the power series for $f(z)$

$$\mathrm{Log}(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \dots.$$

The power series on the RHS converges for $|z| < 1$.
Now if we plug in $z = -1$ then the RHS becomes

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots.$$

This converges, as it is the alternating harmonic series.
But it we plug in $z = 1$ then the RHS becomes

$$-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \dots = -\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right).$$

This diverges, as the harmonic series diverges.
To get an example that converges at i and not at $-i$ let

$$g(z) = f(iz).$$

7. As

$$\frac{z}{z+1}$$

is holomorphic except at -1 it follows that

$$\left(\frac{|z|}{|z+1|}\right)^{3/2}$$

is continuous, except at -1. Thus $f(z)$ is locally bounded, except at -1.

By Riemann's boundedness theorem, it follows that $f(z)$ has removable singularities except at -1. Thus $f(z)$ extends to a holomorphic function except possibly at -1.

Consider $g(z) = (z+1)^2 f(z)$. As the only possible singularity of $f(z)$ is the number -1 it follows that $g(z)$ has at most one isolated singularity at -1.

Now

$$\begin{aligned} |g(z)| &= |(z+1)^2 f(z)| \\ &= |z+1|^2 |f(z)| \\ &\leq |z+1|^2 \left(\frac{|z|}{|z+1|}\right)^{3/2} \\ &= |z+1|^{1/2} |z|^{3/2}. \end{aligned}$$

Thus $g(z)$ is bounded near -1 and so $g(z)$ has a removable singularity at -1.

It follows that $g(z)$ extends to an entire function. As

$$\frac{|z|}{|z+1|}$$

approaches 1 as z goes to infinity, it follows that $g(z)$ is bounded. By Liouville's theorem it is constant. But

$$\begin{aligned} |g(0)| &= |1|^{1/2} |0|^{3/2} \\ &= 0. \end{aligned}$$

Thus $g(z)$ is identically zero. But then $f(z)$ is identically zero.