PRACTICE PROBLEMS FOR THE SECOND MIDTERM

1. (a) Give the definition of:
   (i) a power series;
   (ii) the centre of a power series;
   (iii) the radius of convergence of a power series;
   (iv) a (complex) analytic function;
   (v) the Riemann zeta function;
   (vi) (complex) differentiable at a point;
   (vii) a holomorphic function;
   (viii) an entire function;
   (ix) the tangent vector to a curve;
   (x) a conformal map;
   (xi) a line integral;

   (b) State
   (i) the Cauchy-Riemann equations;
   (ii) Green's theorem;
   (iii) Cauchy's theorem;
   (iv) Cauchy's integral formula.

2. Show that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

diverges, whilst

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

converges.

2. Find the first five terms of the power series expansion of

$$\frac{e^z}{1-z}$$

centred at 0. What is the radius of convergence?

3. Where is the following function holomorphic? Find its derivative

$$\frac{e^{2z^2}}{z^2 - 5z + 6}.$$

4. Show that the first quadrant

$$\{\, z \in \mathbb{C} \,|\, 0 < \operatorname{Arg}(z) < \frac{\pi}{2} \,\}$$

and the unit disk are conformally equivalent.

5. Write down the polar form of the Cauchy-Riemann equations and check that the functions

$$u(r,\theta) = r^m \cos(m\theta) \qquad \text{and} \qquad v(r,\theta) = r^m \sin(m\theta)$$

satisfy these equations.

6. Suppose that P and Q are two functions on the annulus

$$V = \{\, z \in \mathbb{C} \mid a < |z| < b \,\}$$

which have continuous partial derivatives. If

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

then show that the integral

$$\int_{\gamma_r} P\,\mathrm{d}x + Q\,\mathrm{d}y$$

is independent of r, where γ_r is the circle of radius $r \in (a,b)$ centred at the origin and we traverse γ_r counterclockwise.