MODEL ANSWERS TO THE NINTH HOMEWORK

1. The function

$$\frac{1}{z^2 - z}$$

is not holomorphic at $z = 0$ and at $z = 1$. There are two relevant circles, the circle of radius 1 centred at -1 and the circle of radius 2 centred at -1. The point $1/2$ is between these two circles, so we want to compute the Laurent series for the annulus

$$U = \{\, z \in \mathbb{C} \,|\, 1 < |z+1| < 2 \,\}.$$

We have

$$\begin{aligned}\frac{1}{z^2 - z} &= \frac{1}{z(z-1)} \\ &= \frac{1}{z-1} - \frac{1}{z}.\end{aligned}$$

The first function is holomorphic on the disk of radius 2 centred at -1. The second function is holomorphic on the region $|z+1| > 1$ and is zero at infinity. We have

$$\begin{aligned}\frac{1}{z-1} &= \frac{1}{-2 + z + 1} \\ &= -\frac{1}{2}\frac{1}{1-(z+1)/2} \\ &= -\frac{1}{2} - \frac{z+1}{4} - \frac{(z+1)^2}{8} + \dots.\end{aligned}$$

This is a power series centred at $z = -1$ with radius of convergence 2. For the second function we have

$$\begin{aligned}-\frac{1}{z} &= -\frac{1}{-1 + (z+1)} \\ &= \frac{1}{z+1}\frac{1}{1-1/(z+1)} \\ &= \frac{1}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \dots.\end{aligned}$$

Thus

$$\frac{1}{z^2-z} = \frac{1}{z-1} - \frac{1}{z}$$
$$= \cdots + \frac{1}{(z+1)^3} + \frac{1}{(z+1)^2} + \frac{1}{z+1} - \frac{1}{2} - \frac{z+1}{4} - \frac{(z+1)^2}{8} + \dots.$$

is the Laurent expansion.

(b) The function

$$\frac{z-1}{z+1}$$

is not holomorphic at $z=-1$. There are two relevant circles, the circle of radius 0 centred at -1 and the circle of radius ∞ centred at -1. The point $1/2$ is between these two circles, so we want to compute the Laurent series for the annulus

$$U = \{\, z \in \mathbb{C} \,|\, 0 < |z+1| < \infty \,\}.$$

We have

$$\frac{z-1}{z+1} = \frac{z+1-2}{z+1}$$
$$= -\frac{2}{z+1} + 1.$$

This is a Laurent expansion and so it is the Laurent expansion.

2. (a) We have

$$\frac{1}{z+z^2} = \frac{1}{z(z+1)}.$$

This has a simple pole at $z=0$.

$$\operatorname{Res}_0 \frac{1}{z+z^2} = \lim_{z\to 0} \frac{1}{z+1}$$
$$= 1.$$

(b) We have

$$z\cos\left(\frac{1}{z}\right) = z\left(1 - \frac{1}{2z^2} + \frac{1}{4!z^4} + \dots\right)$$
$$= z - \frac{1}{2z} + \frac{1}{4!z^3} + \dots.$$

It follows that the residue at $z=0$ is $-1/2$.

(c) As $\sinh z$ has a simple zero at 0 it follows that

$$\frac{\sinh z}{z^4(1-z^2)}$$

has a pole of order 3 at 0. We could try multiplying by z^3 and differentiating twice to get the residue; this doesn't seem to work very well.
If we expand the power series for $\sinh z$ and for the reciprocal of $1 - z^2$ we get:

$$\frac{\sinh z}{z^4(1-z^2)} = \frac{1}{z^4}\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots\right)(1 + z^2 + z^4 + \dots)$$

We want the coefficient of $1/z$ after we multiply out. So we want the coefficient of z^3 from the second two expressions. This is

$$1 + \frac{1}{3!} = \frac{7}{6}.$$

Therefore the residue is $7/6$.
3. As $(z-a)^n f(z)$ is bounded near a and it has an isolated singularity at a, it follows that $(z-a)^n f(z)$ has a removable singularity at $z = a$. In particular there is a holomorphic function $g(z)$ such that

$$(z-a)^n f(z) = g(z).$$

Suppose that $g(z)$ has a zero of order m at a. Then there is a function $h(z)$ holomorphic at a such that

$$g(z) = (z-a)^m h(z) \qquad \text{and} \qquad h(a) \neq 0.$$

It follows that

$$(z-a)^n f(z) = (z-a)^m h(z).$$

If $m \geq n$ then

$$f(z) = (z-a)^{m-n} h(z)$$

has a removable singularity at a. Otherwise

$$f(z) = \frac{h(z)}{(z-a)^{n-m}}.$$

In this case $f(z)$ has a pole order at most n.
4. (a) We use the residue theorem. The function

$$\frac{z}{\cos z}$$

has isolated singularities at

$$z = \pm\pi/2$$

which are both inside the circle of radius 2, as $\pi/2 < 2$. $\cos z$ has simple singularities at these points and so the function

$$\frac{z}{\cos z}$$

has simple poles at $\pm\pi/2$. To compute the residue at $\pi/2$ we multiply by $(z-\pi/2)$ and take a limit:

$$\begin{aligned}\operatorname{Res}_{\pi/2}\frac{z}{\cos z} &= \lim_{z\to\pi/2}\frac{z(z-\pi/2)}{\cos z}\\ &= \lim_{z\to\pi/2}\frac{2z-\pi/2}{-\sin z}\\ &= -\frac{\pi}{2}.\end{aligned}$$

To compute the residue at $-\pi/2$ we multiply by $(z+\pi/2)$ and take a limit:

$$\begin{aligned}\operatorname{Res}_{-\pi/2}\frac{z}{\cos z} &= \lim_{z\to-\pi/2}\frac{z(z+\pi/2)}{\cos z}\\ &= \lim_{z\to-\pi/2}\frac{2z+\pi/2}{-\sin z}\\ &= -\frac{\pi}{2}.\end{aligned}$$

Now we apply the residue theorem:

$$\begin{aligned}\oint_{|z|=2}\frac{z}{\cos z}\,\mathrm{d}z &= 2\pi i\operatorname{Res}_{\pi/2}\frac{z}{\cos z}+2\pi i\operatorname{Res}_{-\pi/2}\frac{z}{\cos z}\\ &= 2\pi i\left(-\frac{\pi}{2}-\frac{\pi}{2}\right)\\ &= -2\pi^2 i.\end{aligned}$$

(b) The function

$$\frac{e^{-z}}{z^2}$$

has an isolated singularity at zero, which is inside the circle. As the function has a double pole at 0, to compute the residue we multiply by z^2 and differentiate once:

$$\begin{aligned}\operatorname{Res}_0\frac{e^{-z}}{z^2} &= \lim_{z\to 0}-e^{-z}\\ &= -1.\end{aligned}$$

Now we apply the residue theorem:

$$\begin{aligned}\oint_{|z|=3}\frac{e^{-z}}{z^2}\,\mathrm{d}z &= 2\pi i\operatorname{Res}_0\frac{e^{-z}}{z^2}\\ &= -2\pi i.\end{aligned}$$

(c) The function

$$z^2 e^{1/z}$$

has a singularity at $z = 0$. As we have an essential singularity we simply have to compute the Laurent series:

$$z^2 e^{1/z} = z^2 \left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots\right)$$
$$= z^2 + z + \frac{1}{2} + \frac{z}{6} + \dots.$$

Thus the residue is

$$\mathrm{Res}_0\, z^2 e^{1/z} = \frac{1}{6}.$$

Now we apply the residue theorem:

$$\oint_{|z|=1} z^2 e^{1/z}\, \mathrm{d}z = 2\pi i\, \mathrm{Res}_0\, z^2 e^{1/z}$$
$$= \frac{\pi i}{3}.$$