MODEL ANSWERS TO THE EIGHTH HOMEWORK

1. By assumption the image of f is constrained to lie in a vertical strip,

$$a < \mathrm{Re}(w) < b.$$

We can map this into the unit circle, using the standard tricks. If we translate by $-a$, we may assume that the image of f lies in the vertical strip

$$0 < \mathrm{Re}(w) < b.$$

In particular it lies in the right hand plane. Rotating by i we may assume that the image lies in the upper half plane. Applying a Möbius transformation, as in Homework ??, we may assume that the image lies in the unit circle. It follows that the composition is constant by Liouville's theorem. But then f is constant.
Or we could compose with the exponential map and use the same argument.
2. We follow the proof of Liouville's theorem. By assumption there is a real number M_0 such that

$$|f(z)| \leq M_0|z^n| \qquad \text{for} \qquad |z| > R.$$

As $f(z)$ is entire it has a power series expansion whose radius of convergence is ∞,

$$f(z) = \sum_k a_k z^k.$$

The coefficients are given by Cauchy's formula

$$a_k = \frac{1}{2\pi i}\oint_{|z|=r} \frac{f(z)\,\mathrm{d}z}{z^{k+1}},$$

where the radius is any positive real number r. We estimate the absolute value of a_k.
The circle of radius r centred at the origin has length

$$L = 2\pi r.$$

We also have

$$\begin{aligned}
\left|\frac{f(z)}{z^{k+1}}\right| &= \frac{|f(z)|}{|z^{k+1}|} \\
&= \frac{|f(z)|}{r^{k+1}} \\
&\le \frac{M_0 r^n}{r^{k+1}} \\
&= \frac{M_0}{r^{k+1-n}}.
\end{aligned}$$

(16.2) implies that

$$\begin{aligned}
|a_k| &= \left|\frac{1}{2\pi i}\oint_{|z|=r}\frac{f(z)\,\mathrm{d}z}{z^{k+1}}\right| \\
&\le LM \\
&\le 2\pi r \frac{M_0}{2\pi r^{k+1-n}} \\
&= \frac{M_0}{r^{k-n}}.
\end{aligned}$$

As r tends to infinity the last quantity tends to zero if $k > n$. The only possibility is that if $k > n$ then

$$|a_k| = 0 \qquad \text{so that} \qquad a_k = 0.$$

Thus

$$f(z) = a_0$$

is a constant.

3. Expand the following functions in power series about ∞:

(a)

$$\frac{1}{z^2+1} = \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6} - \frac{1}{z^8} + \dots,$$

see example 17.4.

(b) We have

$$\begin{aligned}
g(w) &= f\left(\frac{1}{w}\right) \\
&= \frac{(1/w)^2}{(1/w)^3+1} \\
&= \frac{w}{1+w^3}
\end{aligned}$$

As g is holomorphic at 0 it follows that the original function is holomorphic at ∞. We have

$$\begin{aligned} \frac{w}{1+w^3} &= w(1 + w^3 + w^6 + w^9 + \dots) \\ &= w + w^4 + w^7 + w^{10}. \end{aligned}$$

Thus

$$\frac{z^2}{z^3-1} = \frac{1}{z} + \frac{1}{z^4} + \frac{1}{z^7} + \frac{1}{z^{10}} + \dots$$

is the power series expansion at ∞.

(c) We have

$$\begin{aligned} g(w) &= f\left(\frac{1}{w}\right) \\ &= e^{w^2}. \end{aligned}$$

As g is holomorphic at 0 it follows that the original function is holomorphic at ∞. We have

$$e^{w^2} = 1 + w^2 + \frac{w^4}{2} + \frac{w^6}{3!} + \dots,$$

Thus

$$e^{1/z^2} = 1 + \frac{1}{z^2} + \frac{1}{2z^4} + \frac{1}{6z^6} + \dots$$

is the power series expansion at ∞.

(d) We have

$$\begin{aligned} g(w) &= f\left(\frac{1}{w}\right) \\ &= \frac{\sinh w}{w}. \end{aligned}$$

g is holomorphic at 0 by the usual argument. The power series for

$$\sinh w$$

has no constant term and so it is divisible by w. It follows that the original function is holomorphic at ∞. We have

$$\begin{aligned} \frac{\sinh w}{w} &= \frac{1}{w}\left(w + \frac{w^3}{+}\frac{w^5}{5!} + \dots\right) \\ &= 1 + \frac{w^2}{3!} + \frac{w^4}{5!} + \dots. \end{aligned}$$

Thus

$$z\sinh(1/z) = 1 + \frac{1}{6z^2} + \frac{1}{5!z^4} + \frac{1}{7!z^6} + \dots$$

is the power series expansion at ∞.

4. Let

$$g(u) = f\left(\frac{1}{u}\right)$$

We have to show g is holomorphic at 0. We have

$$\begin{aligned}
g(u) &= \iint_E \frac{\mathrm{d}x\mathrm{d}y}{\frac{1}{u} - z} \\
&= \iint_E u \frac{\mathrm{d}x\mathrm{d}y}{1 - uz} \\
&= u \iint_E \frac{\mathrm{d}x\mathrm{d}y}{1 - uz} \\
&= u \iint_E \left(1 + uz + u^2 z^2 + \dots\right) \mathrm{d}x\mathrm{d}y \\
&= u \iint_E 1 \,\mathrm{d}x\mathrm{d}y + u^2 \iint_E z \,\mathrm{d}x\mathrm{d}y + u^3 \iint_E z^2 \,\mathrm{d}x\mathrm{d}y + \dots.
\end{aligned}$$

This gives us a power series in u

$$\sum c_n u^n,$$

with coefficients

$$c_n = \iint_E z^{n-1} \,\mathrm{d}x\mathrm{d}y.$$

5. (a)

$$\frac{z^2 + 1}{z^2 - 1}$$

is zero at $\pm i$. To determine the order, we only need to worry about the numerator. Both zeroes are simple, as the derivative of $z^2 + 1$ is z and this is not zero at ± 1. Or we could use the fact that

$$z^2 + 1 = (z - i)(z + i),$$

so that both roots are visibly simple.

(b)

$$\frac{1}{z} + \frac{1}{z^5} = \frac{z^4 + 1}{z^5}$$

is zero when

$$z^4 = -1.$$

These are given by the four primitive eighth roots of unity,

$$\omega = e^{i\pi/4}; \qquad \omega^3 = e^{3i\pi/4}; \qquad \omega^5 = e^{5i\pi/4} \qquad \text{and} \qquad \omega^7 = e^{7i\pi/4}.$$

These are all simple zeroes. As before, the denominator does not change the order of the zeroes and the derivative of $z^4 + 1$ is $3z^3$ which is non-zero at all of the roots.

(c)

$$z^2 \sin z$$

is zero when

$$z = m\pi$$

is an integer multiple of π. The order of zeroes of a product is the sum of the order of zeroes. z^2 has a double zero at 0 and $\sin z$ has simple zeroes at the integer multiples of π. Thus $z^2 \sin z$ has simple zeroes at the non-zero integer multiples of π and has a zero of order three at 0.

Challenge Problems: (Just for fun)

6. The coefficients are either 0 or 1. The sequence to compute the limsup is therefore the constant sequence

$$1, \qquad 1, \qquad 1, \qquad 1, \ldots.$$

The nth root of 1 is 1, the reciprocal of 1 is 1 and so the radius of convergence is 1.
Now pick a root of unity ω. Suppose that $\omega^m = 1$. If $n \geq m$ then m divides $n!$ so that $n! = m \cdot d$ for some integer d. We have

$$\begin{aligned} \omega^{n!} &= \omega^{md} \\ &= (\omega^m)^d \\ &= 1^d \\ &= 1. \end{aligned}$$

Thus

$$\begin{aligned} f(r\omega) &= \sum_n (r\omega)^{n!} \\ &= \sum_n r^{n!}\omega^{n!} \\ &= \sum_n r^{n!}\omega^{n!} \\ &= \sum_{n<m} r^{n!}\omega^{n!} + \sum_{n\geq m} r^{n!}\omega^{n!} \\ &= r^{n!}(\sum_{n<m} \omega^{n!}) + \sum_{n\geq m} r^{n!}. \end{aligned}$$

Note that we can bring r outside of the first sum, as it is a finite sum. In the limit as r approaches one from below, the first term approaches the sum

$$\sum_{n<m} \omega^{n!}.$$

The second term is at least any partial sum, as the terms are negative. The limit of a partial sum is the number of terms in the partial sum. It follows that the second term clearly diverges, it approaches ∞.

7. Suppose that the centre of the disk is a and the radius is one. Shifting by $-a$ we may assume that the centre of the disk is 0. Multiplying by $1/\rho$ we may assume that we have the unit disk. Composing with $1/z$ we may assume that we miss the outside of the unit disk, not the inside. But to say we miss the outside of the unit disk is to say that the image lies in the unit disk. Therefore we have a bounded function g (bounded by one). g is the composition of f with a Möbius transformation T. Therefore g is constant by Louiville's theorem. As T has an inverse, it follows that f is constant.

Note that this easily implies (1), since the complement of a strip contains a lot of open disks.