MODEL ANSWERS TO THE SEVENTH HOMEWORK

1. We have to compute the following limit (if it exists at all)

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}.$$

As a first step let us manipulate the numerator.

$$\begin{aligned}
f(z) - f(a) &= \int_\gamma \frac{h(t)}{t - z} \, \mathrm{d}t - \int_\gamma \frac{h(t)}{t - a} \, \mathrm{d}t \\
&= \int_\gamma \frac{h(t)}{t - z} - \frac{h(t)}{t - a} \, \mathrm{d}t \\
&= \int_\gamma \frac{h(t)(t - a) - h(t)(t - z)}{(t - z)(t - a)} \, \mathrm{d}t \\
&= \int_\gamma \frac{h(t)(z - a)}{(t - z)(t - a)} \, \mathrm{d}t \\
&= (z - a) \int_\gamma \frac{h(t)}{(t - z)(t - a)} \, \mathrm{d}t.
\end{aligned}$$

If we divide through by $z - a$ we get

$$\int_\gamma \frac{h(t)}{(t - z)(t - a)} \, \mathrm{d}t.$$

If we take the limit as z approaches a we get

$$\int_\gamma \frac{h(t)}{(t - a)^2} \, \mathrm{d}t$$

(this is a uniform limit as a is at least a fixed distance from γ). It follows that the limit exists, so that f is a holomorphic function and the derivative at a is

$$\int_\gamma \frac{h(t)}{(t - a)^2} \, \mathrm{d}t.$$

2. (a) As 1 belongs to the open disk centred at 0 of radius 2 and z^n is entire, if we apply Cauchy's integral formula then we get

$$\begin{aligned}
\oint_{|z|=2} \frac{z^n}{z - 1} \, \mathrm{d}z &= 2\pi i 1^n \\
&= 2\pi i.
\end{aligned}$$

(b) The function

$$\frac{z^n}{z-2}$$

is holomorphic on the open disk of radius $3/2$, which includes the closed unit disk, so that Cauchy's Theorem implies

$$\oint_{|z|=1} \frac{z^n}{z-2}\,\mathrm{d}z = 0.$$

(c) As $\sin z$ is entire, by Cauchy's integral formula we get

$$\begin{aligned}\oint_{|z|=1} \frac{\sin z}{z}\,\mathrm{d}z &= 2\pi i \sin 0\\ &= 0.\end{aligned}$$

(d) As $\cosh z$ is holomorphic and the second derivative of $\cosh z$ is $\cosh z$ we get

$$\begin{aligned}\oint_{|z|=1} \frac{\cosh z}{z^3}\,\mathrm{d}z &= \frac{2\pi i}{2!}\cosh 0\\ &= \pi i.\end{aligned}$$

(e) There are two cases. If $m \leq 0$ then

$$\frac{e^z}{z^m} = z^{-m}e^z,$$

is entire, so that the integral is zero by Cauchy's theorem. If $m > 0$ then we have to compute the $(m-1)$th derivative of e^z at 0, which is 1 and divide by $(m-1)!$. Putting this together we get

$$\oint_{|z|=1} \frac{e^z}{z^m}\,\mathrm{d}z = \begin{cases} \frac{2\pi i}{(m-1)!} & \text{if } m > 0\\ 0 & \text{if } m \leq 0.\end{cases}$$

(f) First note that the distance of 0 to $1+i$ is

$$\sqrt{2} > \frac{5}{4}.$$

Therefore the open disk of radius $5/4$ centred at $1+i$ contains no non-positive real numbers. In particular $\operatorname{Log} z$ is a holomorphic function on an open set containing the closed disk of radius $5/4$ centred at $1+i$. The derivative of $\operatorname{Log} z$ is $1/z$. As 1 belongs to the disk of radius $5/4$ centred at $1+i$, we have

$$\begin{aligned}\oint_{|z-1-i|=5/4} \frac{\operatorname{Log} z}{(z-1)^2}\,\mathrm{d}z &= \frac{2\pi i}{1}\frac{1}{1}\\ &= \pi i.\end{aligned}$$

(g) Note that

$$\frac{1}{(z^2-4)e^z} = \frac{e^{-z}}{(z^2-4)}$$

is holomorphic on the open disk of radius 2 centred at the origin. It's derivative is

$$\frac{-e^{-z}(z^2-4) - e^{-z}2z}{(z^2-4)^2} = e^{-z}\frac{4-2z-z^2}{(z^2-4)^2}.$$

We have

$$\begin{aligned}\oint_{|z|=1} \frac{\mathrm{d}z}{z^2(z^2-4)e^z} &= \frac{2\pi i}{1}e^0\frac{4}{4^2} \\ &= \frac{\pi i}{2}.\end{aligned}$$

(h) The circle centred at 1 with radius 2 contains both 0 and 2 but not -2. There are two obvious ways to deal with the fact that the integrand is not defined at two points of the open disk of radius 2 centred at 1. The first is to use partial fractions to split the integrand into two pieces, one with a denominator that vanishes at 0 and the other that vanishes at 2 and then integrate both pieces separately.
We have

$$\frac{1}{z^2(z^2-4)} = \frac{A}{z^2} + \frac{B}{(z^2-4)}.$$

This gives

$$1 = A(z^2-4) + Bz^2.$$

It follows that

$$A = -\frac{1}{4} \qquad \text{and} \qquad B = \frac{1}{4}.$$

Thus

$$\begin{aligned}\oint_{|z-1|=2} \frac{\mathrm{d}z}{z^2(z^2-4)e^z} &= \frac{1}{4}\oint_{|z-1|=2} \frac{e^{-z}\mathrm{d}z}{(z^2-4)} - \frac{1}{4}\oint_{|z-1|=2} \frac{e^{-z}\mathrm{d}z}{z^2} \\ &= \frac{1}{4}2\pi i\frac{e^{-2}}{4} - \frac{1}{4}\frac{2\pi i}{1} - e^0 \\ &= \frac{1}{8}\pi i e^{-2} + \frac{1}{2}\pi i.\end{aligned}$$

The second way to deal with the fact that the denominator is zero at two numbers is to use Cauchy's theorem. The disk of radius 2 centred at 1 contains two disks of radius 1/2, one centred at 0 and the other centred at 2. If we remove both of these disks the resulting region U has boundary the circle of radius 2 and the two circles of radius

1/2 centred at 0 and 2, but with the opposite orientation. Cauchy's theorem impies that

$$\int_{\partial U} \frac{\mathrm{d}z}{z^2(z^2-4)e^z} = 0.$$

It follows that

$$\oint_{|z-1|=2} \frac{\mathrm{d}z}{z^2(z^2-4)e^z} = \oint_{|z|=1/2} \frac{\mathrm{d}z}{z^2(z^2-4)e^z} + \oint_{|z-2|=1/2} \frac{\mathrm{d}z}{z^2(z^2-4)e^z}.$$

The first integral we computed in (g), as the integral around a circle of radius 1/2 or 1 is the same. For the second integral we have

$$\begin{aligned}\oint_{|z-2|=1/2} \frac{\mathrm{d}z}{z^2(z^2-4)e^z} &= 2\pi i \frac{1}{2^2}\frac{1}{2+2}e^{-2} \\ &= 2\pi i \frac{1}{2^2}\frac{1}{2+2}e^{-2} \\ &= \frac{1}{8}\pi i e^{-2}.\end{aligned}$$

Putting this together we get

$$\oint_{|z-1|=2} \frac{\mathrm{d}z}{z^2(z^2-4)e^z} = \frac{\pi i}{2} + \frac{1}{8}\pi i e^{-2},$$

the same as before.
3. The rectangle with vertices $\pm R$ and $\pm R + it$ has four sides,

$$\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4,$$

where γ_1 is the horizontal line from $-R$ to R, γ_2 is the vertical line from R to $R + it$, γ_3 is the horizontal line from $R + it$ to $-R + it$, and γ_4 is the vertical line from $-R + it$ to $-R$.
The function $e^{-z^2/2}$ is holomorphic inside the rectangle bounded by γ and so

$$\oint_\gamma e^{-z^2/2}\,\mathrm{d}z = 0,$$

by Cauchy's integral formula. Note that if $t < 0$ our choice of orientation is the reverse orientation to normal. However if you flip the sign of zero, you still get zero.

The length of the two paths γ_2 and γ_4 is t, which is independent of R. On both γ_2 and γ_4 we have $x = \pm R$ and $|y| \leq |t|$, and so

$$\begin{aligned} |e^{-z^2/2}| &= e^{-x^2/2+y^2/2} \\ &= e^{-x^2/2}e^{y^2/2} \\ &= e^{-R^2/2}e^{y^2/2} \\ &\leq e^{-R^2/2}e^{t^2/2}. \end{aligned}$$

Thus

$$\begin{aligned} \left| \int_{\gamma_2+\gamma_4} e^{-z^2/2}\, \mathrm{d}z \right| &\leq \int_{\gamma_2+\gamma_4} |e^{-z^2/2}|\, \mathrm{d}z \\ &\leq \int_{\gamma_2+\gamma_4} |e^{-z^2/2}|\, \mathrm{d}z \\ &\leq 2le^{t^2/2}e^{-R^2/2}. \end{aligned}$$

As R goes to infinity $2le^{t^2/2}e^{-R^2/2}$ goes to zero.
For the path γ_1, we use the parametrisation $\gamma_1(s) = s$, $s \in [-R, R]$. We have

$$\begin{aligned} \lim_{R\to\infty} \int_{\gamma_1} e^{-z^2/2}\, \mathrm{d}z &= \lim_{R\to\infty} \int_{-R}^{R} e^{-x^2/2}\, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} e^{-x^2/2}\, \mathrm{d}x \\ &= \sqrt{2\pi}. \end{aligned}$$

For the path γ_3, we use the parametrisation $\gamma_3(s) = -s + it$, $s \in [-R, R]$. We have

$$\begin{aligned} \lim_{R\to\infty} \int_{\gamma_3} e^{-z^2/2}\, \mathrm{d}z &= -\lim_{R\to\infty} \int_{-R}^{R} e^{-(x+it)^2/2}\, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} e^{-x^2/2-xit+t^2/2}\, \mathrm{d}x \\ &= e^{t^2/2} \int_{-\infty}^{\infty} e^{-x^2/2}e^{-itx}\, \mathrm{d}x \end{aligned}$$

Putting all of this together we get

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}e^{-itx}\, \mathrm{d}x = e^{-t^2/2}.$$

4. The Cauchy integral formula says that

$$f(a) = \frac{1}{2\pi i} \oint_{|z-a|=\rho} \frac{f(z)}{z-a}\,\mathrm{d}z.$$

We compute the RHS using the parametrisation

$$\gamma(\theta) = a + \rho e^{i\theta} \qquad \text{where} \qquad \theta \in [0, 2\pi].$$

We get

$$\begin{aligned}
\frac{1}{2\pi i} \oint_{|z-a|=\rho} \frac{f(z)}{z-a}\,\mathrm{d}z &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho e^{\theta}} i\rho e^{i\theta}\,\mathrm{d}\theta \\
&= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho e^{\theta}} i\rho e^{i\theta}\,\mathrm{d}\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} f(a+\rho e^{i\theta})\,\mathrm{d}\theta.
\end{aligned}$$

Taking the real parts of both sides of the first equality gives

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a+\rho e^{i\theta})\,\mathrm{d}\theta.$$

Challenge Problems: (Just for fun)

4. (continued). Suppose that a is maximum of u, so that $u(z) \leq m = u(a)$. Then

$$\begin{aligned}
m &= u(a) \\
&= \frac{1}{2\pi} \int_0^{2\pi} u(a+\rho e^{i\theta})\,\mathrm{d}\theta \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} m\,\mathrm{d}\theta \\
&= m.
\end{aligned}$$

It follows that the inequality is in fact an equality. But then

$$u(a+\rho e^{i\theta}) = m$$

all the way around the circle, since the integral computes the average value of $u(z)$ on the circle. Thus $u(z) = m$ for any point on any circle in U centred at a. Thus $u(z) = m$ on any disk centred at a. It follows that $u(z) = m$ on any disk in U centred at a point b where $u(b) = m$. It is not hard to conclude that $u(z) = m$ for every $z \in U$, so that u is constant.

Note that $-u$ is the real part of the holomorphic function $-f$. If u has a minimum then $-u$ has a maximum and so $-u$ is constant. But then u is constant.
5. We have

$$\begin{aligned}
2\pi i &= \oint_{|z|=R} \frac{1}{z}\,\mathrm{d}z \\
&= \oint_{|z|=R} \frac{p(z)}{zp(z)}\,\mathrm{d}z \\
&= \oint_{|z|=R} \frac{p(0)}{zp(z)}\,\mathrm{d}z + \oint_{|z|=R} \frac{q(z)}{p(z)}\,\mathrm{d}z \\
&= \oint_{|z|=R} \frac{p(0)}{zp(z)}\,\mathrm{d}z \\
&= p(0) \oint_{|z|=R} \frac{1}{zp(z)}\,\mathrm{d}z.
\end{aligned}$$

To get the first equality we applied Cauchy's integral formula. To get the penultimate equality we applied Cauchy's theorem to the rational function

$$\frac{q(z)}{p(z)},$$

which is holomorphic as $p(z)$ has no zeroes.
We now have to estimate

$$zp(z)$$

on a big circle. Note that

$$\frac{1}{p(z)}$$

goes to zero, as the radius R of the circle goes to infinity. The length of the circle goes like $2\pi R$. Cancelling of R we still get an upper bound that goes to zero. This is not possible as $2\pi i$ is not zero.