MODEL ANSWERS TO THE FOURTH HOMEWORK

1. Let $s_1, s_2, \dots$ be the partial sums of the alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \dots$$

and let $t_1, t_2, \dots$ be the partial sums of the series

$$t = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \dots.$$

We look at groups of four terms of the first series and compare them with three terms of the second series:

$$\begin{aligned}
(t_{3n} - t_{3(n-1)}) - (s_{4n} - s_{4(n-1)})& \\
&= \left(\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n}\right) - \left(\frac{1}{4n-3} - \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{1}{4n}\right) \\
&= \frac{1}{4n-2} - \frac{1}{4n} \\
&= \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n}\right) \\
&= \frac{1}{2}(s_{2n} - s_{2(n-1)}).
\end{aligned}$$

Summing from 1 to n, we get a lot of cancelling and we get

$$t_{3n} - s_{4n} = \frac{1}{2} s_{2n}.$$

Taking the limit as n goes to ∞ we get

$$\begin{aligned}
t - s &= \lim_{n\to\infty} t_{3n} - \lim_{n\to\infty} s_{4n} \\
&= \lim_{n\to\infty} (t_{3n} - s_{4n}) \\
&= \lim_{n\to\infty} \frac{1}{2} s_{2n} \\
&= \frac{1}{2} \lim_{n\to\infty} s_{2n} \\
&= \frac{1}{2} s,
\end{aligned}$$

so that

$$t - s = \frac{s}{2}.$$

Thus

$$t = \frac{3s}{2}.$$

2. We compare the first series with the integral

$$\int_1^\infty \frac{1}{x \ln x} \, \mathrm{d}x.$$

The sum

$$\sum_{n=2}^{m-1} \frac{1}{n \ln n}$$

can be interpreted as a Riemann sum for the integral

$$\int_2^m \frac{1}{x \ln x} \, \mathrm{d}x$$

which is greater than the integral. We can evaluate the integral by subtitution:

$$\begin{aligned} \int_2^m \frac{1}{x \ln x} \, \mathrm{d}x &= \int_{\ln 2}^{\ln m} \frac{1}{u} \, \mathrm{d}u \\ &= \Big[\ln u \Big]_{\ln 2}^{\ln m} \\ &= \ln \ln m - \ln \ln 2. \end{aligned}$$

Note this goes to infinity as m goes to infinity (really, really slowly). As the integral diverges, so does the sum.
We compare the second series with the integral

$$\int_1^\infty \frac{1}{x \ln^2 x} \, \mathrm{d}x.$$

The sum

$$\sum_{n=3}^{m} \frac{1}{n \ln^2 n}$$

can be interpreted as a Riemann sum for the integral

$$\int_2^m \frac{1}{x \ln x} \, \mathrm{d}x$$

which is less than the integral. We can evaluate the integral by subtitution:

$$\begin{aligned}\int_2^m \frac{1}{x\ln^2 x}\,\mathrm{d}x &= \int_{\ln 2}^{\ln m} \frac{1}{u^2}\,\mathrm{d}u \\ &= \left[-\frac{1}{u}\right]_{\ln 2}^{\ln m} \\ &= \frac{1}{\ln 2} - \frac{1}{\ln m}.\end{aligned}$$

Now the second term goes to zero, as m goes to infinity. Thus the integral converges and so does the sum.

3. (a) We start with the standard power series for e^z and subtitute z with $2z$:

$$e^{2z} = 1 + 2z + 2z^2 + \frac{4z^3}{3} + \frac{2^4 z^4}{4!} + \dots$$

The radius of convergence is half of infinity, that is, infinity.

(b) We take linear combinations of the power series

$$\begin{aligned}\cos z &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots,\\ \sin z &= z - \frac{z^3}{3!} + \dots,\end{aligned}$$

to get

$$\begin{aligned}2\cos z - 3\sin z &= 2(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots) - 3(z - \frac{z^3}{3!} + \dot{)}\\ &2 - 3z - z^2 + \frac{z^3}{2} + \frac{2z^4}{4!} + \dots.\end{aligned}$$

The radius of convergence is ∞.

(c) We start with the power series

$$\sin z = z - \frac{z^3}{3!} + \dots$$

and substitute z^2 for z

$$\sin z^2 = z^2 - \frac{z^6}{3!} + \dots.$$

The radius of convergence is ∞.

(d) We have

$$\frac{1}{3-2z} = \frac{1/3}{1-2/3z}.$$

We take the power series for the geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + \dots$$

substitute $2z/3$ for z and then multiply the result by $1/3$:

$$\begin{aligned}\frac{1}{3-2z} &= \frac{1/3}{1-2/3z} \\ &= \frac{1}{3} + \frac{2z}{9} + \frac{4z^2}{27} + \dots.\end{aligned}$$

The radius of convergence is $3/2$.
(e) We take the power series for the geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + \dots$$

and substitute z^2 for z

$$\frac{1}{1-z^2} = 1 + z^2 + z^4 + z^6 + \dots.$$

The radius of convergence is 1.
(f) We could divide by 6 and substitute something of the form $az+bz^2$ for z in the geometric series.
It is easier to simply to use the method of partial fractions

$$\frac{2z-5}{6-5z+z^2} = \frac{A}{3-z} + \frac{B}{2-z}.$$

We get

$$2z - 5 = A(2-z) + B(3-z).$$

Plugging in $z = 3$ we see $B = 1$ and so $A = 1$.
We get

$$\begin{aligned}\frac{2z-5}{6-5z+z^2} &= \frac{1}{3-z} + \frac{1}{2-z} \\ &= \frac{1/3}{1-z/3} + \frac{1/2}{1-z/2} \\ &= \frac{1}{3} + \frac{z}{9} + \frac{z^2}{27} + \dots + \frac{1}{2} + \frac{z}{4} + \frac{z^2}{8} + \dots \\ &= \frac{5}{6} + \frac{13z}{36} + \left(\frac{1}{27} + \frac{1}{8}\right) z^2 + \dots.\end{aligned}$$

The radius of convergence is at least 2, the minimum of 2 and 3. But the LHS is not defined at $z = 2$ and so the radius of convergence is at most 2.
The radius of convergence is 2.

(g) We have

$$\begin{aligned}\frac{1}{1-z} &= \frac{1}{1-i-(z-i)}\\ &= \frac{1/(1-i)}{1-(z-i)/(1-i)}\\ &= \frac{1}{1-i} + \frac{z-i}{(1-i)^2} + \frac{(z-i)^2}{(1-i)^3} + \dots .\end{aligned}$$

The radius of convergence is the radius of convergence of the geometric series

$$\frac{1}{1-z} = 1 + z + z^2 + \dots$$

divided by the reciprocal of the modulus of $1-i$, that is, multiplied by the modulus of $1-i$. The modulus of $1-i$ is $\sqrt{2}$ and so the radius of convergence is $\sqrt{2}$.
Indeed the centre of convergence is i and the function

$$\frac{1}{1-z}$$

is not defined at $z = 1$, whose distance to i is $\sqrt{2}$.

Challenge Problems: (Just for fun)

4. The gradient of $xy = a$ at the point (x_0, y_0) is orthogonal to the tangent line at the point (x_0, y_0) and the gradient of $x^2 - y^2 = b$ at the point (x_1, y_1) is orthogonal to the the tangent line at the point (x_1, y_1). So we just have to show that the gradients at the same point are orthogonal. The gradient of $xy = a$ at the point (x, y) is (y, x) and the gradient of $x^2 - y^2 = b$ at the point (x, y) is $(2x, -2y)$. As the dot product

$$\begin{aligned}(y, x) \cdot (2x, -2y) &= 2xy - 2xy\\ &= 0,\end{aligned}$$

the two curves are orthogonal.
5. We have

$$\begin{aligned}s_{2(n+1)} &= s_{2n} + \frac{1}{2n+1} - \frac{1}{2n+2}\\ &> s_{2n}\end{aligned}$$

and

$$\begin{aligned}s_{2(n+1)+1} &= s_{2n+1} - \frac{1}{2n+2} + \frac{1}{2n+3}\\ &< s_{2n+1}.\end{aligned}$$

On the other hand

$$\begin{aligned} s_{2n+1} &= s_{2n} - \frac{1}{2n+1} \\ &> s_{2n}. \end{aligned}$$

It follows that

$$s_2 < s_4 < s_6 < \cdots < s_5 < s_3 < s_1.$$

The even terms are bounded from above and increasing so that they tend to a limit s_e. The odd terms are bounded from below and decreasing so that they tend to a limit s_o. It is clear that $s_e < s_o$. But the difference between s_{2n} and s_{2n+1} is decreasing so that $s_e = s_0$. This is then the common limit s.