MODEL ANSWERS TO THE THIRD HOMEWORK

1. (a) Clear.
(b) Note that the real line, the line $y = 0$, contains 0, 1 and ∞. The image of the real line must contain -1, 1 and i. But the line connecting 1 to -1 is the real line and this doesn't contain i.
(c) Suppose that the image of the real line is a line or a circle. Using (b) it must be a circle. Now the unit circle contains ± 1 and i. But there is at most one circle through three points p_1, p_2 and p_3. Indeed the centre of the circle must lie on the perpendicular bisector of p_1 and p_2 and on p_2 and p_3. As two lines meet in a point, this determines the centre of the circle and then this determines the radius.
(d) We have to check that if z is real then $M(z)$ has modulus one. We have

$$\begin{aligned} |M(x)|^2 &= \left| \frac{2ix + 1 - i}{2x - 1 + i} \right|^2 \\ &= \frac{1 + (2x-1)^2}{(2x-1)^2 + 1} \\ &= 1. \end{aligned}$$

2. Since ∞ goes to 2 the ratio between a and c is 2. It follows that neither a not c is zero. Dividing through by c, we may assume that $c = 1$ and $a = 2$, so that we have something of the form

$$z \longrightarrow \frac{2z + b}{z + d}$$

Since 0 goes to 1 the ratio between b and d is 1 so that we we have something of the form

$$z \longrightarrow \frac{2z + b}{z + b}.$$

The condition that 1 goes to $1 + i$ implies that

$$\frac{2 + b}{1 + b} = 1 + i.$$

Thus

$$2 + b = (1 + b)(1 + i) \qquad \text{and so} \qquad ib = 2 - i - 1 = 1 - i$$

It follows that $b = -1 - i$.

The Möbius transformation

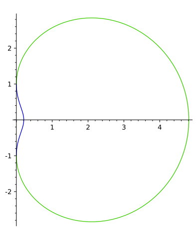
$$z \longrightarrow \frac{2z-1-i}{z-1-i}$$

takes 0 to 1, 1 to $1+i$ and ∞ to 2.
3. We have

$$\begin{aligned}\overline{e^z} &= \overline{e^{x+iy}} \\ &= \overline{e^x e^{iy}} \\ &= e^x \overline{e^{iy}} \\ &= e^x e^{-iy} \\ &= e^{x-iy} \\ &= e^{\bar{z}}.\end{aligned}$$

4. (a) Vertical lines get sent to circles centred at the origin. The line $x=0$ is sent to the unit circle and the line $x=1$ is sent to the circle of radius $e^1 = e$. Thus the image is the annulus

$$\{\, z \in \mathbb{C} \,|\, 1 < |z| < e \,\}.$$



(b) Horizontal lines $y=\theta$ get sent to half lines through the origin with angle θ. The line $y=\theta$ cuts the circle $|z| = \pi/2$ at

$$-\sqrt{\frac{\pi^2}{4}-\theta^2} \qquad \text{and} \qquad \sqrt{\frac{\pi^2}{4}-\theta^2}.$$

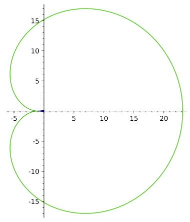
Thus the image of the closed disk of radius $\pi/2$ is sent to the closed set bounded by

$$r = e^{-\sqrt{\frac{\pi^2}{4}-\theta^2}} \qquad \text{and} \qquad r = e^{\sqrt{\frac{\pi^2}{4}-\theta^2}}.$$

where

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$$

This means that the image lies in the half space to the right of the



imaginary axis.

(c) The image of the closed disk of radius π is sent to the closed set bounded by

$$r = e^{-\sqrt{\pi^2-\theta^2}} \qquad \text{and} \qquad r = e^{\sqrt{\pi^2-\theta^2}}.$$

where

$$-\pi \leq \theta \leq \pi.$$

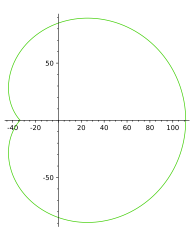
Note that the image of the circle just starts to overlap itself at this point. The two points of the circle $\pm\pi$ get sent to the same point -1.

(d) The image of the closed disk of radius $3\pi/2$ is sent to the closed set bounded by

$$r = e^{-\sqrt{9\pi^2/4-\theta^2}} \qquad \text{and} \qquad r = e^{9\sqrt{\pi^2/4-\theta^2}}.$$

where

$$-\pi \leq \theta \leq \pi.$$



Note that if the angle goes beyond π the circle comes back on itself.

5. (a)

$$\text{Log}(2) = \ln 2.$$

(b)

$$\begin{aligned}\text{Log}(i) &= \ln 1 + i\frac{\pi}{2} \\ &= i\frac{\pi}{2}.\end{aligned}$$

(c)

$$\begin{aligned}\text{Log}(1+i) &= \ln\sqrt{2} + i\frac{\pi}{4} \\ &= \frac{1}{2}\ln 2 + i\frac{\pi}{4}.\end{aligned}$$

(d)

$$\begin{aligned}\operatorname{Log}(1+i\sqrt{3})/2 &= \ln 1 + i\frac{\pi}{3} \\ &= i\frac{\pi}{3}.\end{aligned}$$

Challenge Problems: (Just for fun)

6. (a) It is evident

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 matrix with complex entries and the condition that $ad-bc \neq 0$ is precisely the condition that this matrix is invertible.
(b) Suppose that the matrices A and B are

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

Then the matrix product is

$$AB = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

Now we compute the composition:

$$\begin{aligned}(M \circ N)(z) &= M(N(z)) \\ &= M\left(\frac{ez+f}{gz+h}\right) \\ &= \frac{a\frac{ez+f}{gz+h}+b}{c\frac{ez+f}{gz+h}+d} \\ &= \frac{a(ez+f)+(gz+h)b}{c(ez+f)+(gz+h)d} \\ &= \frac{aez+af+bgz+bh}{cez+cf+dgz+dh} \\ &= \frac{(ae+bg)z+(af+bh)}{(ce+dg)z+(cf+dh)}.\end{aligned}$$

It follows that $M \circ N$ is a Möbius transformation and AB is a matrix corresponding to the product.
(c) The inverse matrix of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{is the matrix} \qquad B = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

If N is the Möbius transformation corresponding to B then the composition of M and N is the Möbius transformation whose matrix is the identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

that is, the identity function. But then M is invertible and the inverse is the Möbius transformation given by the inverse matrix.