MODEL ANSWERS TO THE SECOND HOMEWORK

1. We use DeMoivre's theorem:

$$\begin{aligned}\cos 4\theta + i\sin 4\theta &= (\cos\theta + i\sin\theta)^4\\ &= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta.\end{aligned}$$

Equating real and imaginary part gives

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\\ \sin 4\theta &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta.\end{aligned}$$

2. (a) The identity

$$z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \cdots + z + 1)$$

follows just by multiplying out. See also homework 1.
(b) Suppose that ζ is an nth root of unity and $\zeta \neq 1$.
We have

$$\begin{aligned}0 &= \zeta^n - 1\\ &= (\zeta - 1)(\zeta^{n-1} + \zeta^{n-2} + \cdots + \zeta + 1).\end{aligned}$$

As the first factor is non-zero the second factor must be zero.
3. (a) We saw that

$$e^{iz} = \cos z + i\sin z.$$

Thus

$$\begin{aligned}e^{iz} &= \cos z + i\sin z\\ e^{-iz} &= \cos z - i\sin z.\end{aligned}$$

Adding and subtracting gives

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

(b) This is clear from (1) and the fact that e^{iz} is periodic with period 2π.
(c) We have

$$\begin{aligned}\cos(z+w) + i\sin(z+w) &= e^{i(z+w)}\\ &= e^{iz}e^{iw}\\ &= (\cos z + i\sin z)(\cos w + i\sin w)\\ &= (\cos z\cos w - \sin z\sin w) + i(\cos z\sin w - \sin z\cos w).\end{aligned}$$

Thus

$$\cos(z+w) + i\sin(z+w) = (\cos z\cos w - \sin z\sin w) + i(\cos z\sin w - \sin z\cos w)$$
$$\cos(z+w) - i\sin(z+w) = (\cos z\cos w - \sin z\sin w) - i(\cos z\sin w - \sin z\cos w).$$

Adding and subtracting gives the addition formulas:

$$\begin{aligned}\cos(z+w) &= \cos z\cos w - \sin z\sin w\\ \sin(z+w) &= \cos z\sin w + \sin z\cos w.\end{aligned}$$

4. (a) We have

$$\begin{aligned}\cos z &= \cos(x+iy)\\ &= \cos x\cos(iy) - \sin x\sin iy\\ &= \cos x\cosh y - i\sin x\sinh y.\end{aligned}$$

Similarly

$$\begin{aligned}\sin z &= \sin(x+iy)\\ &= \cos x\sin(iy) + \sin x\cos iy\\ &= \sin x\cosh y + i\cos x\sinh y.\end{aligned}$$

(b) We have

$$\begin{aligned}|\cos z|^2 &= (\cos x\cosh y)^2 + (\sin x\sinh y)^2\\ &= \cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y\\ &= \cos^2 x(1-\sinh^2 y) + \sin^2 x\sinh^2 y\\ &= \cos^2 x + (\cos^2 x + \sin^2 x)\sinh^2 y\\ &= \cos^2 x + \sinh^2 y.\end{aligned}$$

Similarly, we have

$$\begin{aligned}|\sin z|^2 &= (\sin x\cosh y)^2 + (\cos x\sinh y)^2\\ &= \sin^2 x\cosh^2 y + \cos 2x\sinh^2 y\\ &= \sin^2 x(1-\sinh^2 y) + \cos^2 x\sinh^2 y\\ &= \sin^2 x + (\cos^2 x + \sin^2 x)\sinh^2 y\\ &= \sin^2 x + \sinh^2 y.\end{aligned}$$

(c) Note that

$$\cos z = 0 \qquad \text{if and only if} \qquad \cos^2 x + \sinh^2 y = 0.$$

If a sum of squares is zero then each term is zero. If

$$\cos x = 0 \qquad \text{and} \qquad \sinh y = 0,$$

then we have $x = \pi/2$ plus a multiple of π and $y = 0$.

On the other hand

$$\sin z = 0 \qquad \text{if and only if} \qquad \sin^2 x + \sinh^2 y = 0.$$

If

$$\sin x = 0 \qquad \text{and} \qquad \sinh y = 0,$$

then we have x is a multiple of π and $y = 0$.
(d) Suppose that ω is a period of $\sin z$. Then

$$\begin{aligned} \sin \omega &= \sin 0 \\ &= 0. \end{aligned}$$

As the zeroes of the sine function are all real, it follows that ω is real. But then ω is a period of $\sin x$. It follows that ω is a multiple of 2π.
Now suppose that ω is a period of $\cos z$. Then

$$\begin{aligned} \cos \pi/2 + \omega &= \cos \pi/2 \\ &= 0. \end{aligned}$$

As the zeroes of the cosine function are all real, it follows that ω is real. But then ω is a period of $\cos x$. It follows that ω is a multiple of 2π.
5. (a) If

$$z = re^{i\theta} \qquad \text{then} \qquad z^2 = r^2 e^{2i\theta}.$$

Thus the function $z \longrightarrow z^2$ squares the modulus and doubles the argument.
Note that for the first quadrant we have

$$\{ z \in \mathbb{C} \mid \operatorname{Re}(z) > 0, \operatorname{Im}(z) > 0 \} = \{ z \in \mathbb{C} \,|\, 0 < \operatorname{Arg}(z) < \pi/2 \}.$$

and for the upper half plane

$$\mathbb{H} = \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \} = \{ z \in \mathbb{C} \,|\, 0 < \operatorname{Arg}(z) < \pi \}.$$

Since any positive real has a square root, $z \longrightarrow z^2$ establishes a correspondence between the first quadrant and the upper half plane.
(b) Now we want to triple the argument. If

$$z = re^{i\theta} \qquad \text{then} \qquad z^3 = r^3 e^{3i\theta}.$$

Thus the function $z \longrightarrow z^3$ cubes the modulus and triples the argument. As every positive real number is the cube of a real positive number, the function $z \longrightarrow z^3$ establishes a correspondence between the two regions.
(c) If

$$z = re^{i\theta} \qquad \text{then} \qquad z^n = r^n e^{ni\theta}.$$

Thus the function $z \longrightarrow z^n$ raises the modulus to the nth power and multiplies the angle by n. As every positive real number is the nth

power of a real positive number, the function $z \longrightarrow z^n$ establishes a correspondence between the two regions.
(d) If

$$z = re^{i\theta} \qquad \text{then} \qquad 1/z = r^{-1}e^{-i\theta}.$$

Thus the function $z \longrightarrow 1/z$ takes the reciprocal of the modulus and flips the sign of the argument. A Point inside the unit circle has modulus less than one and its reciprocal is a point of modulus greater than one, a point outside the unit circle. As every non-zero positive real is the reciprocal of a positive real number and every real is the negative of another real, $z \longrightarrow 1/z$, maps the region

$$\{\, z \in \mathbb{C} \,|\, 0 < |z| < 1 \,\}$$

that is, the punctured unit disc, to the region

$$\{\, z \in \mathbb{C} \,|\, 1 < |z| \,\}$$

that is, the outside of the unit disc.

Challenge Problems: (Just for fun)

6. As $i = e^{i(2n+1/2)\pi}$, for any integer n, we have

$$\begin{aligned} i^i &= (e^{i(2n+1/2)\pi})^i \\ &= e^{i^2(2n+1/2)\pi} \\ &= e^{-(2n+1/2)\pi}. \end{aligned}$$

As usual the ambiguity in the argument percolates to an ambiguity in taking powers.
Note that i^{i^i} is ambiguous, in just the same way that a^{b^c} is ambiguous. One interpretation is

$$\begin{aligned} i^{i^i} &= (e^{-(2n+1/2+2mi)\pi})^i \\ &= e^{-(2ni+i/2-2m)\pi} \\ &= e^{(-i/2+2m)\pi} \\ &= -ie^{2m\pi}, \end{aligned}$$

where m is any integer.
Another is

$$\begin{aligned} i^{i^i} &= (i)^{e^{-(2n+1/2)\pi}} \\ &= (e^{(2m+1/2)i\pi})^{e^{-(2n+1/2)\pi}} \\ &= e^{(2m+1/2)i\pi e^{-(2n+1/2)\pi}}, \end{aligned}$$

where m and n are arbitrary integers.

7. We could try a Möbius transformation. We want to send three points of the real line to three points of the unit circle.

$$M(z) = \frac{2iz + 1 - i}{2z - 1 + i},$$

has this property. We make sure the upper half plane goes to the inside of the unit disk and not the outside. i is a point of the upper half plane.

$$\begin{aligned} M(i) &= \frac{2ii - 1 - i}{2i - 1 - i} \\ &= \frac{-3 - i}{i - 1}. \end{aligned}$$

The square of the modulus of this number is

$$\frac{1^2 + 3^2}{1^2 + 1^2} = 5 > 1.$$

Thus we get a point outside the unit circle.
There are two ways to fix this. One way is to post-compose with the reciprocal function

$$z \longrightarrow 1/z.$$

This switches the inside of the circle with the outside. This works but then it takes some work to compute the composition (although, composition of Möbius transformations is in fact matrix multiplication).
Another way is to pre-compose. If the upper half plane is sent to the outside of the unit circle then the lower half plane is sent to the inside. The map

$$z \longrightarrow -z$$

switches the upper and lower half planes and so the map

$$z \longrightarrow \frac{-2iz + 1 - i}{-2z - 1 + i},$$

is the Möbius transformation we are looking for
8. One can solve this problem directly. Another way is to break this problem into pieces by writing the Möbius transformation as a composition of Möbius transformations. The first step is to send r to ∞ (if it is not already there). The transformation

$$z \longrightarrow \frac{1}{z - r}$$

has this property. p and q are mapped to two other points, necessarily complex numbers.

Now let us send p to 0 and at the same time fix ∞. Möbius transformations that fix ∞ look like

$$z \longrightarrow az + b.$$

The transformation

$$z \longrightarrow z - p,$$

fixes ∞ and sends p to 0. So now p and r are where we want them and we just have to send q to 1, fixing 0 and ∞. As transformations fixing ∞ look like

$$z \longrightarrow az + b$$

and so transformations that fix 0 and ∞ look like

$$z \longrightarrow az.$$

If we want q to go to 1, we let $a = 1/q$ to get

$$z \longrightarrow z/q.$$

This establishes existence. To prove uniqueness uses a trick. If M_1 and M_2 are two Möbius transformations sending p, q and r to 0, 1 and ∞ then the composition

$$M_2 \circ M_1^{-1}$$

is a Möbius transformation that sends 0, 1 and ∞ to 0, 1 and ∞. We already know that to fix 0 and ∞ the transformation must be of the form

$$z \longrightarrow az$$

and to fix 1 means $a = 1$. Thus we get the Möbius transformation

$$z \longrightarrow z$$

which is the identity map. As $M_1 \circ M_2^{-1}$ is the identity it follows that $M_1 = M_2$.