MODEL ANSWERS TO THE FIRST HOMEWORK

1. We saw in the previous homework that a circle of radius ρ and centred at the origin is given by the equation

$$|z - a| = \rho.$$

Squaring both sides we get

$$\begin{aligned}\rho^2 &= |z-a|^2 \\ &= (z-a)\overline{(z-a)} \\ &= (z-a)(\bar{z}-\bar{a}) \\ &= z\bar{z} + z\bar{a} - a\bar{z} + a\bar{a} \\ &= |z|^2 + z\bar{a} - a\bar{z} + |a|^2.\end{aligned}$$

But

$$\begin{aligned}2\,\mathrm{Re}(\bar{a}z) &= \bar{a}z + \overline{\bar{a}z} \\ &= \bar{a}z + a\bar{z}.\end{aligned}$$

Putting this together gives the result.

2. (a) We have

$$\begin{aligned}p(i) &= i^3 + i^2 + i + 1 \\ &= -i + 1 + i + 1 \\ &= 0.\end{aligned}$$

(b) There are any number of ways to proceed. $p(z)$ is a real polynomial and so the complex conjugate of i, $-i$ is another root. We might then guess that -1 is the third root.

Aliter: If the other roots are α and β then we know

$$z^3 + z^2 + z + 1 = (z-i)(z-\alpha)(z-\beta).$$

Multiplying out the RHS and equating coefficients gives us

$$-i\alpha\beta = 1 \qquad \text{and} \qquad -i - \alpha - \beta = 1,$$

so that

$$\alpha\beta = i \qquad \text{and} \qquad \alpha + \beta = -1 - i.$$

Thus α and β are the roots of the quadratic polynomial

$$z^2 + (1+i)z - i.$$

Now complete the square or use the quadratic formula.

Aliter: We can do long division and divide the linear factor $z + i$ into the polynomial $p(z)$. We know we won't get a remainder and the quotient is in fact

$$z^2 + (1 + i)z - i.$$

Aliter: If we multiply $p(z)$ by the polynomial $z - 1$ we get the polynomial

$$z^4 - 1.$$

The roots are the fourth roots of unity. i is a fourth root of unity and 1 is a root of $z-1$. What is left are -1 and $-i$ and these are the other roots.

3. We have

$$i = i \qquad i^2 = -1 \qquad i^3 = -i \qquad \text{and} \qquad i^4 = 1.$$

Thus the powers of i are periodic with period 4. i is an nth root of unity if and only if n is divisible by 4.

4. Let $n \geq 1$ be an integer.

(a) There are three ways to proceed. The easiest is to treat z as a variable. It is clear that

$$(1 + z + z^2 + z^3 + \cdots + z^n)(1 - z) = 1 - z^{n+1}$$

and dividing through by $1 - z$ gives the result.

Aliter: We could use induction on n. The result is clear if $n = 0$, since the LHS is 1 and the RHS is

$$\frac{1 - z}{1 - z}.$$

Assume the result for n and let's see what happens for $n + 1$

$$\begin{aligned} 1 + z + z^2 + z^3 + \cdots + z^n + z^{n+1} &= (1 + z + z^2 + z^3 + \cdots + z^n) + z^{n+1} \\ &= \frac{1 - z^{n+1}}{1 - z} + z^{n+1} \\ &= \frac{1 - z^{n+1} + (1 - z)z^{n+1}}{1 - z} \\ &= \frac{1 - z^{n+2}}{1 - z}. \end{aligned}$$

This completes the induction and the proof.

Aliter: We could recognize that we have a geometric series with common ratio z and use the trick of Gauss. Call the sum on the LHS.

Multiplying by z gives us:

$$S = 1 + z + z^2 + z^3 + \cdots + z^n$$
$$zS = \quad z + z^2 + z^3 + \cdots + z^n + z^{n+1}$$

As the expressions on the RHS have so many common terms it makes sense to subtract:

$$(1-z)S = 1 - z^{n+1}.$$

Dividing gives the result.

(b) We apply (a) with $z = e^{i\theta}$. We get

$$1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{ni\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}.$$

Now we equate the real parts. The real part of the LHS is

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta.$$

For the RHS, we first attack the denominator:

$$\begin{aligned} 1 - e^{i\theta} &= e^{i\theta/2}(e^{-i\theta/2} - e^{i\theta/2}) \\ &= -2ie^{i\theta/2}\sin\theta/2. \end{aligned}$$

Note that the reciprocal of $-ie^{i\theta/2}$ is

$$ie^{-i\theta/2}.$$

Thus the RHS is

$$\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} = \frac{ie^{-i\theta/2} - ie^{i(n+1/2)\theta}}{2\sin\theta/2}.$$

Taking the real part we get

$$\frac{\sin\theta/2 + \sin(n+1/2)\theta}{2\sin\theta/2} = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}.$$

5. Multiplying top and bottom by $\cos\theta$ we get

$$\begin{aligned} \left(\frac{1 + i\tan\theta}{1 - i\tan\theta}\right)^n &= \left(\frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}\right)^n \\ &= \left(\frac{e^{i\theta}}{e^{-i\theta}}\right)^n \\ &= \left(e^{2i\theta}\right)^n \\ &= e^{i2n\theta} \\ &= \frac{1 + i\tan n\theta}{1 - i\tan n\theta}. \end{aligned}$$

To get from the penultimate line to the last line we use the identity established to get from the first line to the third line.

6. We want to solve

$$z^6 = -64.$$

If we put

$$z = re^{i\theta}$$

then we get the equation:

$$r^6 e^{6i\theta} = -64.$$

Taking the modulus of both sides we get

$$r = 2.$$

Cancelling we are reduced to solving:

$$e^{6i\theta} = -1 = e^{i\pi}.$$

One solution is

$$6\theta = \pi \qquad \text{so that} \qquad \theta = \frac{\pi}{6}.$$

But we might go once around the circle so that another solution is

$$6\theta = \pi + 2\pi \qquad \text{so that} \qquad \theta = \frac{\pi}{2}.$$

Continuing in this way gives us all six solutions;

$$\begin{aligned} 6\theta &= 5\pi \qquad \text{so that} \qquad \theta = \frac{5\pi}{6} \\ 6\theta &= 7\pi \qquad \text{so that} \qquad \theta = \frac{7\pi}{6} \\ 6\theta &= 9\pi \qquad \text{so that} \qquad \theta = \frac{3\pi}{2} \\ 6\theta &= 11\pi \qquad \text{so that} \qquad \theta = \frac{11\pi}{6}. \end{aligned}$$

The sixth roots of -1 are therefore

$$e^{i\pi/6}; \quad e^{i\pi/2}; \quad e^{5i\pi/6}; \quad e^{7i\pi/6}; \quad e^{3i\pi/2}; \quad \text{and} \quad e^{11i\pi/6}.$$

The sixth roots of -64 are

$$2e^{i\pi/6}; \quad 2e^{i\pi/2}; \quad 2e^{5i\pi/6}; \quad 2e^{7i\pi/6}; \quad 2e^{3i\pi/2}; \quad \text{and} \quad 2e^{11i\pi/6}.$$

There is an interesting connection between this problem and the problem of finding the twelth roots of unity. If

$$\zeta = e^{i\pi/6}$$

then the powers of ζ are 12th roots of unity. The even powers are sixth roots of unity but the odd powers are sixth roots of -1. Thus we just want the odd powers of ζ:

$$\zeta; \qquad \zeta^3; \qquad \zeta^5; \qquad \zeta^7; \qquad \zeta^9; \qquad \text{and} \qquad \zeta^{11}.$$

7. We first put $1-\sqrt{3}i$ into polar form

$$1-\sqrt{3}i = 2e^{i2\pi/3}.$$

It follows that

$$\begin{aligned}(1-\sqrt{3}i)^{10} &= (2e^{-i2\pi/3})^{10} \\ &= 2^{10}e^{-i20\pi/3} \\ &= 2^{10}e^{-4i\pi/3} \\ &= 2^{10}e^{i2\pi/3} \\ &= 2^9(-1+\sqrt{3}i).\end{aligned}$$

Challenge Problems: (Just for fun)

8. Suppose that $z+w=re^{i\theta}$. We have

$$\begin{aligned}|z+w| &= r \\ &= e^{-i\theta}(z+w) \\ &= \mathrm{Re}(e^{-i\theta}(z+w)) \\ &= \mathrm{Re}(e^{-i\theta}z) + \mathrm{Re}(e^{-i\theta}w) \\ &\leq |z|+|w|.\end{aligned}$$

Note that we get equality if and only if

$$\mathrm{Re}(e^{-i\theta}z) = |z| \qquad \text{and} \qquad \mathrm{Re}(e^{-i\theta}w) = |w|.$$

This happens only if both

$$e^{-i\theta}z \qquad \text{and} \qquad e^{-i\theta}w$$

are real. But then w and z are real scalar multiples of each other and for equality this multiple has to be non-negative.