MODEL ANSWERS TO THE ZEROTH HOMEWORK

1. (a) We have $|z - 1 - i| = 1$ if and only if the distance of z from the point $1 + i$ is one. Thus the set of points satisfying the equation

$$|z - 1 - i| = 1$$

represents a circle of radius one centred around the point $1 + i$.
(b) Note that we have

$$1 < |2z - 6| < 2 \qquad \text{if and only if} \qquad 1/2 < |z - 3| < 1.$$

Thus the set of points satisfying the inequalities

$$1 < |2z - 6| < 2$$

represents the region between two circles, one of radius 1/2 the other of radius 1, both centred at 3. This region is called an annulus.
(c) There is nothing for it but to do some algebra. If $z = x + iy$ then

$$\begin{aligned} |z - 1|^2 + |z + 1|^2 &= (x - 1)^2 + y^2 + (x + 1)^2 + y^2 \\ &= 2x^2 + 2 + 2y^2. \end{aligned}$$

Thus we want the set of points satisfying the inequality

$$x^2 + y^2 < 3.$$

This is the a circle of radius $\sqrt{3}$ centred at the origin. Classically these are known as the circles of Apollonius.
(d) One can proceed as in part (c).
Here is another way. Suppose we replace the inequality by an equality.

$$|z - 1| + |z + 1| = 2.$$

Then we are looking at the set of points in the plane such that the sum of the distances from ± 1 is 2.
Imagine taking a piece of string of length and fixing its endpoints to lie at ± 1. If we stretch the piece of string so there is no slack with a pen, then the curve we trace out is an ellipse with foci at $pm1$ (this is a piece of classical Greek geometry).
In our case the length of the string is the distance between the foci. In this csae the string can only trace out the interval $[-1, 1]$ on the real line.
The region is the empty set.
(e) Let us again start by replacing the inequality with an equality:

$$|z - 1| = |z|.$$

We want the set of points whose distance from the points 0 and 1 is equal. We want the vertical line through the point 1/2:

$$x = 1/2.$$

(f) The set of points satisfying the inequalities

$$0 < \operatorname{Im} z < \pi$$

is the horizontal strip between the two horizontal lines, $y = 0$ and $y = \pi$.
(g) The set of points satisfying the inequalities

$$-\pi < \operatorname{Re} z < \pi$$

is the vertical strip between the two vertical lines, $x = -\pi$ and $x = \pi$.
(h) Let us start by replacing the inequality with an equality:

$$|\operatorname{Re} z| = |z|.$$

We want the distance to the origin to be the same as the absolute value of the real part. As

$$z = \operatorname{Re} z + i \operatorname{Im} z,$$

the triangle inequality implies that $\operatorname{Im} z = 0$. Thus we get the real line. The set of points satisfying the inequality

$$|\operatorname{Re} z| < |z|.$$

is everything but the real line.
(i) We have

$$\begin{aligned} \operatorname{Re}(iz + 2) &= \operatorname{Re}(iz) + 2 \\ &= -\operatorname{Im}(z) + 2. \end{aligned}$$

This gives

$$\operatorname{Im}(z) < 2,$$

the region below the horizontal line $y = 2$.
(j) There is nothing for it but to do some algebra. If $z = x + iy$ then

$$\begin{aligned} |z - i|^2 + |z + i|^2 &= x^2 + (y-1)^2 + x^2 + (y+1)^2 \\ &= 2x^2 + 2 + 2y^2. \end{aligned}$$

Thus we want the set of points satisfying the inequality

$$x^2 + y^2 < 0.$$

We have the empty set.
2. Put $z = x + iy$ and $w = s + it$.

(a) We have

$$\begin{aligned}
\overline{z+w} &= \overline{(x+s)+i(y+t)} \\
&= (x+s)-i(y+t) \\
&= (x-iy)+(s-it) \\
&= \bar{z}+\bar{w}.
\end{aligned}$$

(b) We have

$$\begin{aligned}
\overline{zw} &= \overline{xs-yt+i(xt+ys)} \\
&= xs-yt-i(xt+ys).
\end{aligned}$$

By contrast

$$\begin{aligned}
\bar{z}\bar{w} &= (x-iy)(s-it) \\
&= xs-yt-i(xt+ys).
\end{aligned}$$

(c) We have

$$\begin{aligned}
|\bar{z}| &= |x-iy| \\
&= \sqrt{x^2+y^2} \\
&= |z|.
\end{aligned}$$

(d) We have

$$\begin{aligned}
z\bar{z} &= (x+iy)(x-i) \\
&= x^2+y^2 \\
&= |z|^2.
\end{aligned}$$

3. Let us translate so that z_0 is the origin. Now rotate so that z_1 has no imaginary part. If we square both sides then we get

$$|z|^2 = \rho^2|z-z_1|^2.$$

Let $z = x+iy$ and $z_1 = a$, a real number, We get

$$x^2+y^2 = \rho^2(x-a)^2+\rho^2y^2.$$

Rearranging gives

$$(1-\rho^2)x^2+(1-\rho^2)y^2+2\rho^2ax-\rho^2a^2 = 0.$$

Dividing through by $1-\rho^2$ gives

$$x^2+2\sigma ax+\sigma a^2+y^2 = 0 \qquad \text{where} \qquad \sigma = \frac{\rho^2}{1-\rho^2}.$$

Completing the square gives

$$(x-\sigma)^2+y^2 = \sigma(1-\sigma)a^2.$$

This represents a circle.
If $\rho = 1$ we get the bisector of the two points z_0 and z_1.

Challenge Problems: (Just for fun)

4. (i) as the sum of continuous function is continuous and the product of continuous functions is continuous, it suffices to show that

$$x^m e^{-1/x^2}$$

is continuous at the origin, where m is an integer. Replacing x by $1/y$ it suffices to show that the limit as y goes to infinity of

$$y^m e^{-y^2} = \frac{y^m}{e^{y^2}}$$

exists (here we flipped the sign of m). This is an easy application of L'Hôpital's rule.
(ii) f is certainly differentiable outside the origin. At the origin we go back to the definition of the derivative:

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{p(h) e^{-1/h^2}/h^m}{h} \\ &= \lim_{h \to 0} \frac{p(h) e^{-1/h^2}}{h^{m+1}} \\ &= 0, \end{aligned}$$

by (a). Thus the limit exists and the derivative is zero.
(iii) If we just apply the usual rules of differentiation away from zero then we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{p(x)}{x^m} e^{-1/x^2}\right) &= \frac{p'(x)x^m - mp(x)x^{m-1}}{x^{2m}} e^{-1/x^2} + \frac{p(x)}{x^m}\frac{-2}{x^3} e^{-1/x^2} \\ &= \left(\frac{p'(x)x^m - mp(x)x^{m-1}}{x^{2m}} - \frac{2p(x)}{x^{m+3}}\right) e^{-1/x^2} \\ &= \left(\frac{p'(x)x^m - mp(x)x^{m-1} - 2p(x)x^{m-3}}{x^{2m}}\right) e^{-1/x^2} \\ &= \frac{q(x)}{x^n} e^{-1/x^2}, \end{aligned}$$

where

$$q(x) = p'(x)x^m - mp(x)x^{m-1} - 2p(x)x^{m-3} \qquad \text{and} \qquad n = 2m.$$

(iv) $f(x)$ is infinitely differentiable by induction and the derivatives at the origin are all zero, by (iii).
(v) Just use the usual formula to compute the coefficients of the MacLaurin series.