SECOND MIDTERM
MATH 120A, UCSD, WINTER 20

You have 50 minutes.

There are 5 problems, and the total number of points is 55. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:______________________________

Signature:______________________________

Student ID #:______________________________

Section instructor:______________________________

Section Time:______________________________

Problem	Points	Score
1	15	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total	55	

1. (15pts) (i) *Give the definition of the radius of convergence of a power series.*

The radius of convergence of a power series centred at a is the smallest real number R such that if $|z-a| > R$ then the series always diverges.

(ii) Give the definition of (complex) *differentiable at a point.*

We say that a function $f\colon U \longrightarrow \mathbb{C}$ on a region U is differentiable at a if the limit

$$\lim_{z \to a} \frac{f(z)-f(a)}{z-a}$$

exists.

(iii) *Write down the Cauchy-Riemann equations.*

If u and v are two functions on a region U whose partial derivatives exist then the Cauchy-Riemann equations say

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x.$$

2. (10pts) *Show that the series*

$$\sum_{n=2}^{\infty} \frac{1}{n \log^2 n}$$

converges.

We compare the series with the integral

$$\int_1^{\infty} \frac{1}{x \ln^2 x} \, \mathrm{d}x.$$

The sum

$$\sum_{n=3}^{m} \frac{1}{n \ln^2 n}$$

can be interpreted as a Riemann sum for the integral

$$\int_2^{m} \frac{1}{x \ln x} \, \mathrm{d}x$$

which is less than the integral. We can evaluate the integral by subtitution:

$$\begin{aligned} \int_2^m \frac{1}{x \ln^2 x} \, \mathrm{d}x &= \int_{\ln 2}^{\ln m} \frac{1}{u^2} \, \mathrm{d}u \\ &= \left[-\frac{1}{u} \right]_{\ln 2}^{\ln m} \\ &= \frac{1}{\ln 2} - \frac{1}{\ln m}. \end{aligned}$$

Now the second term goes to zero, as m goes to infinity. Thus the integral converges and so does the sum.

3. (10pts) (i) *Write down the first five terms of the power series of*

$$\frac{\cos(5z^2-4z)}{1-2z}$$

centred at 0.

We have

$$\frac{1}{1-2z} = 1 + 2z + 4z^2 + 8z^3 + 16z^4 + \dots.$$

We also have

$$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{24} + \dots$$

so that

$$\cos(z(5z-4)) = 1 - \frac{z^2(5z-4)^2}{2} + \frac{z^4(5z-4)^4}{24} + \dots$$

Multiplying out gives

$$\begin{aligned}
\frac{\cos 5z^2 - 4z}{1-2z} &= \left(1 + 2z + 4z^2 + 8z^3 + 16z^4 + \dots\right)\left(1 - \frac{z^2(5z-4)^2}{2} + \frac{z^4(5z-4)^4}{24} + \dots\right) \\
&= 1 + 2z + (4-8)z^2 + (8-16+20)z^3 + 16z^4 - 32z^4 + 40z^4 - \frac{25}{2}z^4 + \frac{4^4}{24}z^4 + \dots \\
&= 1 + 2z - 4z^2 + 12z^3 + \left(24 - \frac{25}{2} + \frac{32}{3}\right)z^4 + \dots \\
&= 1 + 2z - 4z^2 + 32z^3 + \left(24 - \frac{11}{6}\right)z^4 + \dots \\
&= 1 + 2z - 4z^2 + 32z^3 + \frac{133}{6}z^4 + \dots.
\end{aligned}$$

(ii) *What is the radius of convergence?*

The power series for the numerator converges everywhere but the power series for the denominator converges for $|z| < 1/2$. But the function is not defined at $z = 1/2$ and so the radius of convergence is $1/2$.

4. (10pts) (i) *Let*

$$h\colon [0,1] \longrightarrow \mathbb{C}$$

be a continuous complex valued function defined on the unit interval $[0,1]$*. Define a function*

$$f\colon U \longrightarrow \mathbb{C} \qquad \text{by the rule} \qquad f(z) = \int_0^1 \frac{h(t)}{t-z}\,\mathrm{d}t,$$

where U *is the region* $\mathbb{C} \setminus [0,1]$*. Show that* f *is holomorphic on* U*.*

We have to compute the following limit (if it exists at all)

$$\lim_{z\to a} \frac{f(z)-f(a)}{z-a}.$$

As a first step let us manipulate the numerator.

$$\begin{aligned} f(z)-f(a) &= \int_0^1 \frac{h(t)}{t-z}\,\mathrm{d}t - \int_0^1 \frac{h(t)}{t-a}\,\mathrm{d}t \\ &= \int_0^1 \frac{h(t)}{t-z} - \frac{h(t)}{t-a}\,\mathrm{d}t \\ &= \int_0^1 \frac{h(t)(t-a)-h(t)(t-z)}{(t-z)(t-a)}\,\mathrm{d}t \\ &= \int_0^1 \frac{h(t)(z-a)}{(t-z)(t-a)}\,\mathrm{d}t \\ &= (z-a)\int_0^1 \frac{h(t)}{(t-z)(t-a)}\,\mathrm{d}t. \end{aligned}$$

If we divide through by $z-a$ we get

$$\int_0^1 \frac{h(t)}{(t-z)(t-a)}\,\mathrm{d}t.$$

If we take the limit as z approaches a we get

$$\int_0^1 \frac{h(t)}{(t-a)^2}\,\mathrm{d}t$$

Thus the limit exists and f is a holomorphic function.

(ii) *What is* $f'(a)$*?*

$$\int_0^1 \frac{h(t)}{(t-a)^2}\,\mathrm{d}t.$$

5. (10pts) *Show that if U is a bounded region with smooth boundary then the area of U is given by the integral*

$$\frac{1}{2i}\int_{\partial U} \bar{z}\,\mathrm{d}z.$$

We want to apply Green's theorem to compute the line integral. If $\gamma = \partial U$ then the integrand of the line integral is

$$\begin{aligned}
\bar{z}\,\mathrm{d}z &= (x - iy)(\mathrm{d}x + i\mathrm{d}y)\\
&= x\mathrm{d}x + y\mathrm{d}y + i(-y\mathrm{d}x + x\mathrm{d}y)\\
&= (x - iy)\mathrm{d}x + (y + ix)\mathrm{d}y\\
&= P\mathrm{d}x + Q\mathrm{d}y.
\end{aligned}$$

Note that

$$\frac{\partial P}{\partial y} = -i \qquad \text{and} \qquad \frac{\partial Q}{\partial x} = i.$$

Green's theorem says

$$\begin{aligned}
\int_\gamma \bar{z}\,\mathrm{d}z &= \int_{\partial U} P\,\mathrm{d}x + Q\,\mathrm{d}y\\
&= \iint_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y\\
&= \iint_U 2i\mathrm{d}x\mathrm{d}y\\
&= 2i\iint_U \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

On the other hand

$$\iint_U \mathrm{d}x\mathrm{d}y$$

is the volume under the graph of the constant function 1, which is the area of U.

Bonus Challenge Problems

6. (10pts) *Let f be a holomorphic function on a region U. Show that if the modulus of f is constant then f is constant.*

As U is connected, we may prove this locally on U. Possibly multiplying f by a constant we may assume f is nowhere real. In this case we can compose with the principal value of the logarithm, to get a holomorphic function

$$g(z) = \mathrm{Log}(f(z)).$$

If $f(z) = re^{i\theta}$ then

$$g(z) = \ln r + i\theta,$$

where θ is the principal value of the argument. As the modulus of f is constant then r is constant. It follows that the real part of g is constant.

Suppose that $g(z) = u(x, y) + iv(x, y)$. As the real part of g is constant then u is constant and so $u_x = u_y = 0$ on U. As g is holomorphic it satisfies the Cauchy-Riemann equations. But then

$$\begin{aligned} v_y &= u_x \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} v_x &= -u_y \\ &= 0. \end{aligned}$$

It follows that v is constant. Therefore g is constant. But then f is constant.

7. (10pts) *Let u be the real part of a holomorphic function f on a region U. Show that if u achieves its maximum then u is constant.*

The Cauchy integral formula says that

$$f(a) = \frac{1}{2\pi i} \oint_{|z-a|=\rho} \frac{f(z)}{z-a}\,\mathrm{d}z.$$

We compute the RHS using the parametrisation

$$\gamma(\theta) = a + \rho e^{i\theta} \qquad \text{where} \qquad \theta \in [0, 2\pi].$$

We get

$$\begin{aligned}\frac{1}{2\pi i} \oint_{|z-a|=\rho} \frac{f(z)}{z-a}\,\mathrm{d}z &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho e^{\theta}} i\rho e^{i\theta}\,\mathrm{d}\theta \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho e^{\theta}} i\rho e^{i\theta}\,\mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(a+\rho e^{i\theta})\,\mathrm{d}\theta.\end{aligned}$$

Taking the real parts of both sides of the first equality gives

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a+\rho e^{i\theta})\,\mathrm{d}\theta.$$

Suppose that a is maximum of u, so that $u(z) \leq m = u(a)$. Then

$$\begin{aligned}m &= u(a) \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(a+\rho e^{i\theta})\,\mathrm{d}\theta \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} m\,\mathrm{d}\theta \\ &= m.\end{aligned}$$

It follows that the inequality is in fact an equality. But then

$$u(a+\rho e^{i\theta}) = m$$

all the way around the circle, since the integral computes the average value of $u(z)$ on the circle. Thus $u(z) = m$ for any point on any circle in U centred at a. Thus $u(z) = m$ on any disk centred at a. It follows that $u(z) = m$ on any disk in U centred at a point b where $u(b) = m$. It is not hard to conclude that $u(z) = m$ for every $z \in U$, so that u is constant.