FIRST MIDTERM
MATH 120A, UCSD, WINTER 20

You have 50 minutes.

There are 4 problems, and the total number of points is 50. Show all your work. *Please make your work as clear and easy to follow as possible.*

Name:________________________________

Signature:________________________________

Student ID #:________________________________

Section instructor:________________________________

Section Time:________________________________

Problem	Points	Score
1	15	
2	10	
3	10	
4	15	
5	10	
6	10	
Total	50	

1. (15pts) (i) *Give the definition of the principal value of the argument.*

If z is a complex number the principal value of the argument, denoted $\mathrm{Arg}(z)$, is the angle the vector (x, y) makes with the x-axis, with values constrained to lie in the range $(-\pi, \pi]$.

(ii) *Give the definition of an open disk.*

If $a \in \mathbb{C}$ is a complex number and $\epsilon > 0$ is a positive real, the open disk centred at a is the set of complex numbers whose distance to a is less than ϵ.

(iii) *Give the definition of a Möbius transformation.*

Any function

$$M : \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \qquad \text{of the form} \qquad z \longrightarrow \frac{az+b}{cz+d},$$

where a, b, c and d are complex numbers, such that $ad - bc \neq 0$.

2. (10pts) (i) *State DeMoivre's theorem.*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

where θ is a real number and n is positive integer.

(ii) *Find formulas for*

$$\cos 4\theta \qquad \text{and} \qquad \sin 4\theta,$$

involving only $\cos\theta$ *and* $\sin\theta$*.*

We apply DeMoivre's theorem with $n = 4$:

$$\begin{aligned}\cos 4\theta + i\sin 4\theta &= (\cos\theta + i\sin\theta)^4 \\ &= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta.\end{aligned}$$

Equating real and imaginary parts we get

$$\begin{aligned}\cos 4\theta &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta \\ \sin 4\theta &= 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta.\end{aligned}$$

3. (10pts) *Write down a Möbius transformation that takes* 0 *to* 1, 1 *to* $1+i$ *and* ∞ *to* 2.

Since ∞ goes to 2 the ratio between a and c is 2. It follows that neither a not c is zero. Dividing through by c, we may assume that $c = 1$ and $a = 2$, so that we have something of the form

$$z \longrightarrow \frac{2z+b}{z+d}$$

Since 0 goes to 1 the ratio between b and d is 1 so that we we have something of the form

$$z \longrightarrow \frac{2z+b}{z+b}$$

The condition that 1 goes to $1+i$ implies that

$$\frac{2+b}{1+b} = 1+i.$$

Thus

$$2+b = (1+b)(1+i) \qquad \text{and so} \qquad ib = 2-i-1 = 1-i$$

It follows that $b = -1-i$.
The Möbius transformation

$$z \longrightarrow \frac{2z-1-i}{z-1-i}$$

takes 0 to 1, 1 to $1+i$ and ∞ to 2.

4. (15pts) *The functions*

$$\sinh\colon \mathbb{C} \longrightarrow \mathbb{C} \qquad \text{and} \qquad \cosh\colon \mathbb{C} \longrightarrow \mathbb{C}$$

are defined by

$$\cosh(z) = \cos(iz) \qquad \text{and} \qquad \sinh(z) = -i\sin(iz).$$

(a) *Show that*

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

(You may use the addition formulae without proof).

The addition formula for sine reads

$$\sin(z + w) = \cos z \sin w + \sin z \cos w,$$

where z and w are complex numbers.
We have

$$\begin{aligned} \sin z &= \sin(x + iy) \\ &= \cos x \sin(iy) + \sin x \cos iy \\ &= \sin x \cosh y + i \cos x \sinh y. \end{aligned}$$

(b) Show that

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

Note that

$$\begin{aligned}\cosh^2 x + \sinh^2 x &= \cos^2(ix) + \sin^2(ix)\\ &= 1.\end{aligned}$$

It follows that we have

$$\begin{aligned}|\sin z|^2 &= (\sin x \cosh y)^2 + (\cos x \sinh y)^2\\ &= \sin^2 x \cosh^2 y + \cos 2x \sinh^2 y\\ &= \sin^2 x(1 - \sinh^2 y) + \cos^2 x \sinh^2 y\\ &= \sin^2 x + (\cos^2 x + \sin^2 x) \sinh^2 y\\ &= \sin^2 x + \sinh^2 y.\end{aligned}$$

(c) Find all zeroes of the sine function, that is, find all solutions of

$$\sin z = 0.$$

Note that

$$\sin z = 0 \qquad \text{if and only if} \qquad \sin^2 x + \sinh^2 y = 0.$$

Bur a sum of squares is zero if and only if each term is zero. If

$$\sin x = 0 \qquad \text{and} \qquad \sinh y = 0,$$

then we have x is a multiple of π and $y = 0$.
So the zeroes of $\sin z$ are just the integer multiples of π.

Bonus Challenge Problems

5. (10pts) *Prove the triangle inequality*

$$|z + w| \leq |z| + |w|,$$

with equality if and only if either $z = 0$ *or* w *is a positive real scalar multiple of* z*.*

Suppose that $z + w = re^{i\theta}$. We have

$$\begin{aligned} |z + w| &= r \\ &= e^{-i\theta}(z + w) \\ &= \mathrm{Re}(e^{-i\theta}(z + w)) \\ &= \mathrm{Re}(e^{-i\theta}z) + \mathrm{Re}(e^{-i\theta}w) \\ &\leq |z| + |w|. \end{aligned}$$

Note that we get equality if and only if

$$\mathrm{Re}(e^{-i\theta}z) = |z| \qquad \text{and} \qquad \mathrm{Re}(e^{-i\theta}w) = |w|.$$

This happens only if both

$$e^{-i\theta}z \qquad \text{and} \qquad e^{-i\theta}w$$

are real. But then w and z are real scalar multiples of each other and for equality this multiple has to be non-negative.

6. (10pts) *Given three distinct points p, q and r of the extended complex plane (so that p, q and r are either complex numbers or ∞) show that there is a unique Möbius transformation*

$$z \longrightarrow \frac{az+b}{cz+d}$$

taking p to 0, q to 1 and r to ∞.

We break this problem into pieces by writing the Möbius transformation as a composition. The first step is to send r to ∞ (if it is not already there). The transformation

$$z \longrightarrow \frac{1}{z-r}$$

has this property.
Now let us send p to 0 and at the same time fix ∞. Möbius transformations that fix ∞ look like

$$z \longrightarrow az+b.$$

The transformation

$$z \longrightarrow z-p,$$

fixes ∞ and sends p to 0. So now p and r are where we want them and we just have to send q to 1, fixing 0 and ∞. As transformations fixing ∞ look like

$$z \longrightarrow az+b$$

transformations that fix 0 and ∞ look like

$$z \longrightarrow az.$$

If we want q to go to 1, we let $a = 1/q$ to get

$$z \longrightarrow z/q.$$

This establishes existence. Observe that if M_1 and M_2 are two Möbius transformations sending p, q and r to 0, 1 and ∞ then the composition

$$M_2 \circ M_1^{-1}$$

is a Möbius transformation that sends 0, 1 and ∞ to 0, 1 and ∞.
We already know that to fix 0 and ∞ the transformation must be of the form

$$z \longrightarrow az$$

and to fix 1 means $a = 1$. Thus the composition $M_1 \circ M_2^{-1}$ is the identity and so it follows that $M_1 = M_2$.