9. Power series

Definition 9.1. *Let z_0 be a complex number. An infinite sum of the form*

$$\sum a_n(z - z_0)^n$$

where $a_1, a_2, \ldots$ are complex numbers is called a ***power series, centred at*** *z_0.*

Of course power series are useless (at least in analysis) unless we have convergence, since we want a function. The weakest type of convergence we can ask for is convergence at a point, called **pointwise convergence**: we can just ask for the series

$$\sum a_n(z_1 - z_0)^n$$

to converge, where z_1 is another complex number. We then let z_1 vary in the domain U. We could ask for a little bit more and ask for absolute convergence.

We can do one more thing, we can view the partial sum

$$s_n(z) = \sum_{k \leq n} a_k(z - z_0)^k$$

as a function of z (in fact, it is just a polynomial of degree at most n). We can then ask for uniform convergence, instead of pointwise convergence:

Definition 9.2. *Let $U \subset \mathbb{C}$ be a region containing the point z_0.*

We say that the power series

$$\sum a_n(z - z_0)^n$$

converges uniformly *on U to the function $s(z)$ if for every $\epsilon > 0$ there an integer n_0 such that if $n > n_0$ then*

$$|s(z) - s_n(z)| < \epsilon$$

for all $n \neq n_0$ and for every $z \in U$.

The crucial point is that the same natural number n_0 works uniformly for all $z \in U$. Uniform convergence is to pointwise convergence as absolute convergence is to conditional convergence; it is far superior. Accept no other form of convergence.

The main result is the following:

Definition-Theorem 9.3. *Let $\sum a_n(z - z_0)^n$ be a power series.*

There is a quantity, either a non-negative real number or infinity, $R \in [0, \infty]$, called the ***radius of convergence*** *with the following properties:*

(1) If z belongs to the open disk, centred around z_0, of radius R then the power series converges absolutely at z.
(2) If z does not belong to the closed disk, centred around z_0, of radius R then the power series diverges.
(3) If $R' < R$ then the power series converges uniformly on the open centred around z_0, of radius R'.

In words, we have the following. Outside the radius of convergence we have divergence; inside we have absolute convergence and we even have uniform convergence, if we stay away from the boundary.

(9.3) makes no statement about what happens on the boundary. In fact all sorts of behaviour and pathologies are possible here.

It is even possible to identify the radius of convergence:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}}.$$

Let us break this formula down a bit. Not surprisingly the radius of convergence only depends on the magnitude of the coefficients and what happens in the limit as n goes to infinity. The faster the absolute value of the coefficients grows the smaller the radius of convergence, which is why there is a reciprocal.

The limsup is defined as follows. Given a sequence of real numbers $c_1, c_2, \dots$ let $d_1, d_2, \dots$ be the sequence of supremums (usually the maximum) of the tails of the sequence $c_1, c_2, \dots$. The limsup of the sequence $c_1, c_2, \dots$ is the limit of the sequence $d_1, d_2, \dots$; $d_1, d_2, \dots$ is a monotonic non-increasing sequence and so this limit always exists. If the limit of the sequence $c_1, c_2, \dots$ exists then it is equal to the limsup, so that most of the time we can ignore the limsup.

Example 9.4. *For the sequence*

$$0, \quad 1, \quad 0, \quad 1, \dots$$

the sequence of supremums is

$$1, \quad 1, \quad 1, \quad 1, \dots.$$

The original limit does not exist but the limit of the second sequence is 1, so the limsup of the first sequence is 1.

Example 9.5. *For the sequence*

$$1, \quad -1/2, \quad 1/3, \quad -1/4, \dots$$

the sequence of supremums is

$$1, \quad 1/3, \quad 1/3, \quad 1/5, \quad 1/5 \dots.$$

Example 9.6. *Recall that we defined the exponential via a power series:*

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \dots.$$

Obviously the centre is the origin.

Here the coefficients are

$$\frac{1}{n!}.$$

No need to take the absolute value and if we take the reciprocal we get

$$n!.$$

Half of the numbers in this product as at least $n/2$ so that

$$\sqrt[n]{n!} > \sqrt{n}/2.$$

Therefore even after you take nth roots the limit is infinity. So the radius of convergence is infinity. Inside the radius of convergence, that is everywhere, we get absolute convergence and away from infinity we get uniform convergence.

In fact we could have found the radius of convergence just using (9.3). This says we always get divergence if we are further away than R. But we know that the exponential power series works everywhere on the real axis and so the radius of convergence must be infinity.

Once we know that the exponential has a power series expansion with radius of convergence infinity we get the same result for cosine and sine. One way is to compute the limsup. Note that half the coefficients of the sine and cosine are zero. When you compute the limsup we ignore the zero coefficients. It is clear that the radius of convergence is again infinity.

Or we could use the following basic results:

Proposition 9.7. *Let $\sum a_n(z-z_0)^n$ and $\sum b_n(z-z_0)^n$ be two power series, centred around the same point z_0. Suppose the radius of convergence of the first series is R_1 and the radius of convergence of the second series is R_2. Let α be a complex number.*

(1) $\sum a_n(\alpha z - z_0)^n$ is a power series centred at z_0 with radius of convergence $R_1/|\alpha|$.

(2) $\sum (a_n + b_n)(z-z_0)^n$ is a power series centred at z_0 with radius of convergence at least $\min(R_1, R_2)$.

(3) $\sum c_n(z-z_0)^n$ is a power series centred at z_0 with radius of convergence at least $\min(R_1, R_2)$, where

$$c_n = \sum_{k+l} a_k b_l.$$

Note that if you multiply the power series $\sum a_n(z-z_0)^n$ and $\sum b_n(z-z_0)^n$ then the coefficients of the product are $c_0, c_1, c_2, \dots$.

We have

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \text{and} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

It follows that $\cos z$ and $\sin z$ have power series with the same radius of convergence as e^z, as $\pm i$ has modulus one.

The other way to generate power series is to use geometric series.

Theorem 9.8.

$$\frac{a}{1-z} = a + az + az^2 + \dots$$

is a power series with radius of convergence 1*.*

Proof. We saw in the homework that

$$\frac{1-z^{n+1}}{1-z} = 1 + z + z^2 + \dots + z^n.$$

Suppose that $|z| < 1$. Then

$$\lim_{n\to\infty} z^{n+1} = 0.$$

Thus

$$\frac{1}{1-z} = 1 + z + z^2 + \dots,$$

for $|z| < 1$. Now multiply by a.

On the other hand, if $|z| > 1$ then

$$\lim_{n\to\infty} z^{n+1}$$

does not exist, since the modulus goes to infinity. □

Note that the Möbius transformation

$$z \longrightarrow \frac{a}{1-z}$$

sends 1 to ∞. Thus there is a natural reason why the radius of convergence is one. The function

$$\mathbb{C} \longrightarrow \mathbb{C} \qquad \text{given by} \qquad z \longrightarrow \frac{a}{1-z},$$

is not defined at $z = 1$. Note also that the coefficients of the power series are all 1. The absolute value of 1 is 1, the nth root of 1 is 1, the reciprocal of 1 is 1 and the limsup of 1 is 1. Thus (9.8) is consistent with (9.3).

In fact this is no coincidence. To prove (9.3) you simply compare $\sum a_n(z-z_0)^n$ with a geometric series.

Note that the complex numbers explain another conundrum.

Example 9.9. *Consider the real function:*

$$\frac{1}{1+x^2}.$$

Starting with the formula

$$\frac{1}{1-x} = 1 + x + x^2 + \dots,$$

we get the power series expansion

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \dots$$

by a simple substitution, replace $-x$ by x^2. It is clear that this doesn't change the radius of convergence.

On the other hand, it is clear that the power series for

$$\frac{1}{1-x}$$

should have radius of convergence one, since the denominator is zero at $x = 1$. But it is not clear why

$$\frac{1}{1+x^2}$$

should have radius of convergence one, since the denominator is not zero at $x = \pm 1$.

If one replaces x by z everything becomes much clearer. We can extend the power series expansion to the whole unit disk:

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - \dots$$

The denominator is zero at $z = \pm i$, two points of the unit circle. It is just that these points are not real.

For an example of a power series with zero radius of convergence, consider the power series

$$\sum n! z^n$$

centred at the origin. We already say that the limsup is zero so that this power series does not converge anywhere, other than 0.

Definition-Theorem 9.10. *The series*

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges absolutely for $\mathrm{Re}(s) > 1$ *and converges uniformly for* $\mathrm{Re}(s) > s_0 > 1$. *The resulting function is called the* ***Riemann zeta function*** *and is denoted* $\zeta(s)$.

On the other hand, if $s = 1$ we get the harmonic series, which does not converge. In fact if a_0 is any complex number whose real part is greater than one then $\zeta(s)$ has a power series expansion centred at a_0 with radius of convergence $R = |a_0 - 1|$.

The truly amazing fact is that one can use the power series expansion to **analytically continue** the Riemann zeta function to the whole complex plane, except $s = 1$. In terms of real variable this never works. There is no way to get around the fact that the function is not defined at $s = 1$. But in the complex plane, one can simply go around the point $s = 1$.

It is important to realise that there is no simple way to describe $\zeta(s)$ for $\mathrm{Re}(s) \leq 1$. It is certainly not given by

$$\sum_{n=1}^{\infty} \frac{1}{n^s},$$

which diverges.

The Riemann zeta function is probably the most famous function in mathematics.

Conjecture 9.11 (Riemann hypothesis)**.** *Let s be a zero of the Riemann zeta function, so that*

$$\zeta(s) = 0.$$

Then either $\mathrm{Im}(s) = 0$ *or* $\mathrm{Re}(s) = 1/2$*.*

The zeroes on the real line are called the trivial zeroes and are well understood. The other zeroes are not so well understood.

Finally an important definition.

Definition 9.12. *Let $f \colon U \longrightarrow \mathbb{C}$ be a function defined on a region U.*

We say that f is an ***analytic*** *function if for every point $a \in U$ there is an open disk containing a such that f is given by a power series on the disk.*

Note that one important subtlety about this definition. We allow the centre of the power series to change as we vary a. In some sense this is natural, as power series only make sense on disks but most regions aren't disks.