8. Sequence and series

We would like to work with power series, which means that we have to view sequences and series from the perspective of the complex numbers. This really isn't very different to what happens for the real numbers.

It is important to realise the issue we have to face:

Example 8.1. *The alternating harmonic series*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \dots$$

converges, but the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

diverges to infinity.

Recall that for a series to converge, the sequence of partial sums should converge to a limit. One way to show that a series converges or diverges is to compare it with an integral.

Proposition 8.2. *Let s be a positive real.*

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges if and only if $s > 1$.

Proof. Let's first see divergence if $s \leq 1$. I claim that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

diverges to infinity if $s \leq 1$. Note that if $t \leq s$ then

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \leq \sum_{n=1}^{\infty} \frac{1}{n^t}.$$

So actually it is enough to show that

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges. Consider the function

$$f \colon \mathbb{R} \longrightarrow \mathbb{R} \qquad \text{given by} \qquad f(x) = \frac{1}{x}.$$

As $f(x)$ is a decreasing function,

$$\sum_{n=1}^{m-1} \frac{1}{n}$$

is a Riemann sum for the integral

$$\int_1^m \frac{1}{x}\,\mathrm{d}x$$

which is larger than the actual integral. But

$$\begin{aligned} \int_1^m \frac{1}{x}\,\mathrm{d}x &= \Big[\ln x\Big]_1^m \\ &= \ln m - \ln 1 \\ &= \ln m. \end{aligned}$$

Now $\ln m$ approaches infinity as m approaches infinity. So the harmonic series diverges.

We now turn to convergence. We use the same trick but the other way around. Consider the function

$$f\colon \mathbb{R} \longrightarrow \mathbb{R} \qquad \text{given by} \qquad f(x) = \frac{1}{x^s}.$$

As $f(x)$ is a decreasing function,

$$\sum_{n=2}^{m} \frac{1}{n^s}$$

is a Riemann sum for the integral

$$\int_1^m \frac{1}{x^s}\,\mathrm{d}x$$

which is smaller than the actual integral. But

$$\begin{aligned} \int_1^m \frac{1}{x^s}\,\mathrm{d}x &= \left[\frac{1}{-sx^{s-1}}\right]_1^m \\ &= \frac{1}{s} - \frac{1}{sm^{s-1}}. \end{aligned}$$

As $s > 1$ the second fraction converges to zero. But then the integral converges as m goes to infinity, so that the series converges. □

Note that

$$\begin{aligned} \frac{1}{n} - \frac{1}{n+1} &= \frac{1}{n(n+1)} \\ &< \frac{1}{n^2}. \end{aligned}$$

Thus

$$\begin{aligned}
1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\dots &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\dots \\
&= \frac{1}{2}+\frac{1}{12}+\frac{1}{30}+\dots \\
&< \frac{1}{1}+\frac{1}{9}+\frac{1}{25}+\dots,
\end{aligned}$$

which converges, by (8.2).

Thus the harmonic series diverges but the alternating harmonic series converges.

Definition 8.3. *We say that a series $\sum a_n$ is* ***absolutely convergent*** *if $\sum |a_n|$ is a convergent series.*

We say that a series $\sum a_n$ is ***conditionally convergent*** *if it is convergent but not absolutely convergent.*

The alternating harmonic series is conditionally convergent.

Theorem 8.4. *If $\sum a_n$ is conditionally convergent and s is a real number or $\pm\infty$ then we may rearrange the terms so that sum $\sum b_n$ converges to s.*

Proof. We start with some basic obervations. We make two bags (or more formally sets), a positive bag and a negative bag. We put the positive terms into the positive bag and we put the absolute value of the negative terms into the negative bag:

$$\begin{aligned}
P &= \{\, a_n \mid a_n \geq 0 \,\} \\
N &= \{\, -a_n \mid a_n < 0 \,\}.
\end{aligned}$$

(For the alternating harmonic series, P contains the reciprocal of the odd natural numbers and N contains the reciprocals of the even natural numbers). For the time being we assume that none of the terms are zero (the zero terms are an annoying little detail).

As $\sum |a_n|$ diverges, the contents of both bags sums to infinity. It follows that the contents of at least one of the bags is infinite. As $\sum a_n$ converges the only possibility is that the content of both bags is infinite.

Lets order the elements of both bags in a decreasing sequence:

$$p_1 \geq p_2 \geq p_3 \dots \qquad \text{and} \qquad n_1 \geq n_2 \geq n_3 \dots.$$

As $\sum a_n$ is convergent we must have $\lim_{n\to\infty} a_n = 0$, the terms approach zero. It follows that both

$$\lim_{n\to\infty} p_n = 0 \qquad \text{and} \qquad \lim_{k\to\infty} n_k = 0.$$

Let's arrange for the sum $\sum b_n$ to be at least a million. Look at the largest element of the negative bag n_1. Let's suppose it is a billion. As the content of the positive bag is infinite, if we keep pulling selecting terms from the positive bag, we eventually get a partial sum s_m of at least a million plus a billion. At that point we take $-n_1$ from the negative bag. Now the partial sum $s_m - n_1$ is at least a million.

Now we inspect the second element of the negative bag, n_2. If $s_m - n_1 - n_2$ is bigger than a million then we just take n_2 from the bag, so that the partial sum is now $s_m - n_1 - n_2$ and we inspect the third element of the negative bag n_3. We continue in this way until the point when we cannot remove any more numbers from the negative bag without going under a million.

Now we look to see how much is left in the positive bag. We removed much more than a billion. However, since the positive bag contained an infinite amount to begin with, it still contains an infinite amount. So if we keep selecting elements of the positive bag we eventually get to a sum sufficiently bigger than a million than we can select the next element of the negative bag.

If we continue in this way, it is not too hard to see that we converge to a number bigger than a million. In fact if you think about it carefully, the sum will converge to exactly a million. Of course there is nothing special about a million. We can make $\sum b_n$ converge to any limit s.

Finally note that we do exhaust the terms in both bags. Even though we (presumably) pull terms from the positive bag faster than we pull them from the negative bag, it is not hard to see that eventually we pull everything from the negative bag.

There a couple of small details. First off the bags aren't really sets, since we allow duplication. This is a minor point and is easily taken care of.

We also have to deal with the case that some of the terms are zero. We make one more bag, and put the zero terms into this bag. Every third time we pull an element from the bag, we take it from the zero bag, if we are able (that is, if the zero bag is not empty). This never changes the partial sum and we only do this to make sure we really do have a rearrangement of $a_1, a_2, \dots$. □

The moral of (8.4) is that if you rearrange the terms of a conditionally convergent series then the sum won't necessarily stay the same. Fortunately we do have

Theorem 8.5. *Let $\sum a_n$ be an absolutely convergent series.*

If $\sum b_n$ is a rearrangement of the series $\sum a_n$ then $\sum b_n$ converges to the same limit as $\sum a_n$.

In words, you can rearrange the terms of an absolutely convergent series without changing the sum.

Definition 8.6. *Let $a_1, a_2, \dots$ be a sequence of complex numbers.*

*We say that $a_1, a_2, \dots$ **converges to** a, denoted $\lim_{n\to\infty} a_n = a$, if for every $\epsilon > 0$ we can find n_0 such that*

$$|a - a_n| < \epsilon \qquad \textit{for all} \qquad n \geq n_0.$$

In words, $a_1, a_2, \dots$ converges to the complex number a, the sequence eventually belongs to every disk centred at a, no matter how small the disk.

Once we know what it means for a sequence to converge, we can define convergence of a series:

Definition 8.7. *We say that the series $\sum a_n$ **converges to** s if the sequence of partial sum $s_n = \sum_{i\leq n} a_i$ converges to s.*

From here we can define absolute convergence and conditional convergence the same way and we still get (8.5).

Example 8.8. *Let s be a complex number, such that $\operatorname{Re}(s) > 0$. The series*

$$\sum_{n=0}^{\infty} \frac{1}{n^s}$$

converges absolutely for $\operatorname{Re}(s) > 1$.

First of all, we need to interpret n^s. For this we use the principal value of the logarithm. If

$$z = n^s \qquad \text{then} \qquad \log z = s \log n.$$

We take the principal value of the logarithm $\operatorname{Log} n = \ln n$:

$$z = e^{s \ln n}.$$

Now we check absolute convergence. Suppose that

$$s = a + ib.$$

Note that

$$\begin{aligned} n^s &= e^{s \ln} \\ &= e^{(a+ib)\ln n} \\ &= e^{a \ln n} e^{ib \ln n}, \end{aligned}$$

so that

$$\begin{aligned}|n^s| &= |e^{a\log n}| \\ &= n^a \\ &= n^{\mathrm{Re}(s)}.\end{aligned}$$

So if we take absolute values then we get the series

$$\sum \frac{1}{n^{\mathrm{Re}(s)}}.$$

This converges for $\mathrm{Re}(s) > 1$, as we already saw.