7. Some elementary functions revisited

We introduce a couple of simple techniques to get a picture of what complex functions look like.

If

$$f: U \longrightarrow \mathbb{C}$$

is a complex function, it is sometimes convenient to split f into its real and imaginary parts. It is then also customary to split the variable $z = x + iy$ into its real and imaginary parts

$$f(z) = u(x, y) + iv(x, y).$$

u and v are real valued complex functions,

$$u: U \longrightarrow \mathbb{R} \qquad \text{and} \qquad v: U \longrightarrow \mathbb{R}.$$

Here we cheat a little bit and blur the distinction between $\mathbb{C}$ and $\mathbb{R}^2$.

Let us consider the function

$$\mathbb{C} \longrightarrow \mathbb{C} \qquad \text{given by} \qquad z \longrightarrow z^2.$$

If we write $z = x + iy$ then

$$\begin{aligned} z^2 &= (x + iy)^2 \\ &= (x^2 - y^2) + 2ixy. \end{aligned}$$

It follows that

$$u(x, y) = x^2 - y^2 \qquad \text{and} \qquad v(x, y) = 2xy.$$

One can use u and v to get a picture of $z \longrightarrow z^2$. For example if we look at the level curves of u, we get hyperbolae:

$$x^2 - y^2 = \text{cst}.$$

If we look at the level curves of v we get hyperbolae the other way:

$$2xy = \text{cst}.$$

In fact hyperbolae from different families intersect each other at right angles.

Another way to think of $z \longrightarrow z^2$ is in terms of polar coordinates. If $z = re^{i\theta}$ then

$$z^2 = r^2 e^{2i\theta}.$$

The function $z \longrightarrow z^2$ squares the modulus and doubles the angle.

It is intriguing to consider how this function maps the complex plane to the complex plane. It is clear that this map is two to one (every number has two square roots) except at 0.

It is convenient to think of the complex plane as a piece of cloth (or even a piece of dough since we want to stretch). To get a picture of what is going on, we first see what happens to the upper half plane:

$$\mathbb{H} = \{\, z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \,\} = \{\, z \in \mathbb{C} \mid 0 < \operatorname{Arg}(z) < \pi \,\}.$$

Since squaring doubles the angle we get the region

$$\{\, z \in \mathbb{C} \mid 0 < \operatorname{Arg}(z) < 2\pi \,\} = \mathbb{C} - [0, \infty).$$

The only complex numbers we don't get are the non-negative reals, which have argument a multiple of 2π. We take our piece of dough and pull it around the origin until it almost joins up with itself.

Something similar happens to the lower half plane

$$\{\, z \in \mathbb{C} \mid \operatorname{Im}(z) < 0 \,\} = \{\, z \in \mathbb{C} \mid \pi < \operatorname{Arg}(z) < 2\pi \,\}.$$

Now when we double the argument we again get every argument except multiples of 2π, that is, every complex number except the non-negative reals:

$$\{\, z \in \mathbb{C} \mid 2\pi < \operatorname{Arg}(z) < 4\pi \,\} = \mathbb{C} - [0, \infty).$$

One can imagine stacking the two pieces of dough one on top of the other and joining them along the non-negative reals. Another way to picture what is going on is to think of a multi-storey car park with two floors. Given one complex number $z = x + iy$ there are two parking spots, one on each level lying over the same complex number. It is important to realise this is only a picture. For example if you try to go up two levels, you just end up at the bottom level and not one floor up.

This becomes more important in terms of the inverse function. If

$$z = w^2 \qquad \text{then} \qquad w = r^{1/2} e^{i\theta/2}.$$

Just take the square root of the modulus and halve the angle. The problem is that there are two square roots, so w is not really a function of z. θ is only well-defined up to a multiple of 2π and so $\theta/2$ is only defined up to a multiple of π. But

$$e^{i\pi} = -1 \neq 1.$$

Changing the value of θ potentially flips the sign of the square root, as one expects.

In fact one cannot get away from this issue. The inverse function is not defined everywhere, since if you go once around the origin, you get into trouble. Let's imagine starting at the complex number 1. It is natural to take 1 as the square root of 1. As you go around the unit circle and you approach 1 from the bottom, the argument approaches 2π, half the argument approaches π and so the square root will approach the value -1 and not 1.

There is a surprisingly straightforward way to avoid this problem. Let U be the region obtained by deleting the negative real axis $(-\infty, 0]$ (in fact we can discard any half line; it makes sense to delete one of the real half lines and we don't want to dicard the non-negative reals). Then we define a function

$$U \longrightarrow \mathbb{C} \qquad \text{by the rule} \qquad z \longrightarrow r^{1/2}e^{i\,\mathrm{Arg}(z)/2}.$$

We simply choose the principal value of the argument and divide this by two. Since we excluded the negative real axis then we don't allow $\mathrm{Arg}(z) = -\pi$.

We can generalise the picture for $z \longrightarrow z^2$ to $z \longrightarrow z^3$. This function is three to one and now there are three levels to the multi-storey car park. More generally still one can look at $z \longrightarrow z^n$.

However there is a better and more interesting way to proceed.

Let's go back to the exponential function.

$$\mathbb{C} \longrightarrow \mathbb{C} \qquad \text{given by} \qquad z \longrightarrow e^z.$$

If $z = x + iy$ then

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x e^{iy} \\ &= re^{i\theta}. \end{aligned}$$

In this case the modulus is

$$r = e^x$$

and the argument is

$$\theta = y.$$

Let $w = e^z$. If we fix y and let x vary we trace out the half line $\mathrm{Arg}(w) = y$. If we fix x and let y vary then we go around a circle of radius $r = e^x$ infinitely often. Horizontal lines go to half lines through the origin and vertical lines go to circles concentric circles centred at the origin.

Definition 7.1. *Let $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ be a function. We say that f is* ***periodic, with period*** $\omega \in \mathbb{C}$*, if*

$$f(z + \omega) = f(z)$$

for all z.

Then the exponential function is periodic with period $2\pi i$ (and $4\pi i$, $6\pi i$, etc). There are two twists in the definition of the period. We allow complex numbers (and not just real numbers) and we don't worry about the smallest value of the period.

The function $z \longrightarrow e^z$ is infinite to one. Now we have a multi-storey car park with infinitely many levels (or infinitely many pieces of dough stacked one on top of the other, like a helix).

Can we define the logarithm function? If

$$z = e^w \qquad \text{then} \qquad w = \ln r + i\theta.$$

As before this prescription does not really define a function. The problem is again the ambiguity in the argument.

We can define an inverse function, if we again cut the complex plane along the line $(-\infty, 0]$ and take the principal value of the argument:

$$U \longrightarrow \mathbb{C} \qquad \text{given} \qquad \operatorname{Log}(z) = \ln r + iArg(z).$$

By convention $\log z = \ln r + i\theta$ is deliberately ambiguous and we θ is only well-defined up to a multiple of $2\pi i$.

We will try to stick to the following naming convention. ln denotes the usual natural logarithm, a real function. log denotes any solution to an exponential equation. Log denotes the principal value of log.

Note that once we have the logarithm we can use it to define the square root, cube root and so on. If

$$z = w^n \qquad \text{then} \qquad \log z = n \log w.$$

Therefore

$$w = e^{\log z/n}.$$

This prescription is ambiguous, since there are many possible values for $\log z$. But if we take the principal value of the logarithm then

$$w = e^{\operatorname{Log} z/n}$$

does define a function on the complement of the non-positive reals. This is the function which sends 1 to 1.