6. The Riemann sphere

It is sometimes convenient to add a point at infinity ∞ to the usual complex plane to get the extended complex plane.

Definition 6.1. *The* ***extended complex plane****, denoted* $\mathbb{P}^1$*, is simply the union of* $\mathbb{C}$ *and the point at infinity.*

It is somewhat curious that when we add points at infinity to the reals we add two points $\pm\infty$ but only only one point for the complex numbers. It is rare in geometry that things get easier as you increase the dimension.

One very good way to understand the extended complex plane is to realise that $\mathbb{P}^1$ is naturally in bijection with the unit sphere:

Definition 6.2. *The* ***Riemann sphere*** *is the unit sphere in* $\mathbb{R}^3$*:*

$$S^2 = \{\, (x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1 \,\}.$$

To make a correspondence with the sphere and the plane is simply to make a map (a real map not a function). We define a function

$$F \colon S^2 - \{(0, 0, 1)\} \longrightarrow \mathbb{C} \subset \mathbb{R}^3$$

as follows. Pick a point $q = (x, y, z) \in S^2$, a point of the unit sphere, other than the north pole $p = N = (0, 0, 1)$. Connect the point p to the point q by a line. This line will meet the plane $z = 0$,

$$\{\, (x, y, 0) \,|\, (x, y) \in \mathbb{R}^2 \,\}$$

in a unique point r. We then identify r with a point $F(q) = x + iy \in \mathbb{C}$ in the usual way.

F is called **stereographic projection**. One way to think of the construction of r from q is as follows. Imagine a source of light at the point p. An object at q will cast a shadow at the point r of the horizontal plane.

Note that F cannot be extended to the whole of S^2. As you get closer to the north pole the point r wanders out to infinity. It is then natural to extend F to a function

$$F \colon S^2 \longrightarrow \mathbb{P}^1$$

by simply sending the point p to the point at infinity.

Note that stereographic projection has some very nice features. The south pole gets sent to the origin. The equator gets sent to the unit circle. Great circles through the North pole (a great circle through the North pole is the same as circle through the South pole) map to lines through the origin. The top hemisphere maps to the exterior of the unit disc and the bottom hemisphere to the unit disc.

Note that if we think of the extended complex plane in terms of the Riemann sphere then the significance of ∞ disappears. The Riemann sphere is surely homogeneous, all points look the same.

Consider the function

$$z \longrightarrow \frac{1}{z}$$

It is naturally defined on the punctured complex plane, the complex plane minus the origin. It is natural to extend it to the extended complex plane. We send 0 to ∞ and ∞ to zero. In fact, if we want to figure out what is happening at ∞ it is natural to work with $1/z$. More about this later.

Definition 6.3. *Let a, b, c and d be complex numbers, such that $ad - bc \neq 0$.*

The Möbius transformation

$$M : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

is the function

$$z \longrightarrow \frac{az + b}{cz + d}.$$

We have to interpret the meaning of the function M in terms of what happens at infinity. There are two cases.

Suppose first that $c \neq 0$. Suppose that z is a complex number, not equal to $-\frac{d}{c}$. Then

$$\frac{az + b}{cz + d}$$

is a complex number. We send $z = -d/c$ to infinity. On the other hand we send ∞ to the complex number a/c.

Now suppose that $c = 0$. Then we may as well suppose that $d = 1$ and the function

$$z \longrightarrow az + b,$$

is a function from the complex numbers to the complex numbers in the usual way. For the extended complex plane we send ∞ to ∞.

In terms of the Riemann sphere, a typical Möbius transformations is a rotation (although there are other symmetries). Every rotation has an axis and so every rotation has two fixed points, where the axis of the rotation meets the sphere. Not every Möbius transformations has two fixed points. The transformation

$$z \longrightarrow z + 1$$

is just a translation of the complex plane. It fixes ∞ but it does not fix any complex number. A Möbius transformation does take lines and circles to lines and circles.

There is also an interesting connection between the Riemann sphere and topology.

If $X \subset \mathbb{C}$ is a subset then we say that X is **simply connected** if X is path connected and every closed path can be continuously deformed to a constant map, keeping the endpoints fixed (actually this is equivalent to allowing the endpoint to move). Informally, think of the closed path as rubber band. Can you move the rubber band around until it shrinks to a point?

Open and closed disks are simply connected, the upper half plane, angular regions. An annulus is not simply connected.

It is possible but somewhat involved to make the definition of simply connected formal. Fortunately for the complex plane there is an ad hoc way to get around this.

Definition 6.4. *We say that an open subset $U \subset \mathbb{C}$ is **simply connected** if X is path connected and the complement inside the extended complex plane is connected.*

To make sense of being connected in the extended complex plane we use the Riemann sphere.

For example the complement of an annulus has two connected components the smaller disk and the complement of the bigger disk.